Omega HW #6 – More logarithms (Solutions)

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Remember that the *base-b logarithm of a*, denoted $\log_b(a)$, is the unique number c such that $a^c = b$. Logarithms satisfy the three properties:

$$\log_b(mn) = \log_b(m) + \log_b(n)$$

$$\log_b(m^c) = c \cdot \log_b(m)$$

$$\log_a(b) \cdot \log_b(c) = \log_a(c)$$

The *natural logarithm* of a, denoted $\ln(a)$, is the area under the graph of $y = 1/x$ from $x = 1$ to $x = a$. We showed that this function satisfies the first two properties above, and is therefore a logarithm. We then used geometric series to show the following formula holds whenever $|x| < 1$:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots = \sum_{k=1}^{\infty} \frac{x^k(-1)^{k-1}}{k}$$

The last expression here is *sigma-notation*, which is a notation we introduced for more compactly expressing sums of arbitrarily many terms.

1. (2) Calculate the following logarithms by hand to three decimal places of accuracy, or write down an arithmetic expression which does so. (For example, you can write down a fraction of two decimal numbers which gives the answer, or a sum of such fractions. You do not need to do the final divisions and additions by hand.)

   (a) $\ln(1.2)$
   Solution: $\ln(1.2) \approx (0.2) - \frac{0.04}{2} + \frac{0.008}{3} - \frac{0.0016}{4} \approx 0.1823$

   (b) $\ln(2)$ (*Hint:* Do not use the alternating harmonic series, because you would need to use too many terms. Instead, use the fact that $2 = (6/5) \cdot (5/4) \cdot (4/3)$, and write $\ln(2)$ as the sum of three logarithms.)
   Solution: Write $\ln(2) = \ln(6/5) + \ln(5/4) + \ln(4/3)$ and approximate these terms.

$$\ln(6/5) \approx \frac{1}{5} - \frac{1}{50} + \frac{1}{375} - \frac{1}{2500} \approx 0.1823$$

$$\ln(5/4) \approx \frac{1}{4} - \frac{1}{32} + \frac{1}{192} - \frac{1}{1024} \approx 0.2230$$

$$\ln(4/3) \approx \frac{1}{3} - \frac{1}{18} + \frac{1}{81} - \frac{1}{324} + \frac{1}{1215} \approx 0.2879$$

   Thus, $\ln(2) \approx 0.1823 + 0.2230 + 0.2879 \approx 0.6932$.

   (c) $\ln(10)$
   Solution: Since $10 = 2^3 \cdot (5/4)$, we have

$$\ln(10) = 3\ln(2) + \ln(5/4) \approx 3(0.6932) + 0.2230 \approx 2.3026$$

(d) $\log_2(10)$

Solution: $\log_2(10) = \frac{\ln(10)}{\ln(2)} \approx \frac{2.3026}{0.6932} \approx 3.3217$

(e) $\ln(0.9)$

Solution: $\ln(0.9) = \ln(1 - 0.1) \approx -0.1 - \frac{0.01}{2} - \frac{0.001}{3} \approx -0.1053$.

(f) $\log_{10}(180)$

Solution:

$$\log_{10}(180) = 2 + \log_{10}(1.8) = 2 + \log_{10}(1.8) = 2 + \frac{\ln(1.8)}{\ln(10)}$$

$$= 2 + \frac{\ln(2) + \ln(0.9)}{\ln(10)} \approx 2 + \frac{0.6932 - 0.1053}{2.3026} \approx 2.2553$$

2. (1) For each of the following sums, convert it from sigma notation to expanded notation, or vice-versa. (Like last time, you can leave the individual terms as products of parts.)

(a) $\sum_{k=3}^{8} 2k$

Solution: $6 + 8 + 10 + 12 + 14 + 16$

(b) $3 + 6 + 12 + 24 + 48 + 96 + 384$

Solution: $\sum_{k=0}^{6} (3 \cdot 2^k)$

(c) $\sum_{k=1}^{5} (k^2 \cdot 3^k)$

Solution: $(1 \cdot 3) + (4 \cdot 9) + (9 \cdot 27) + (16 \cdot 81) + (25 \cdot 243)$

(d) $1 - x^2 + x^4 - x^6 + \ldots - x^{22}$

Solution: $\sum_{k=0}^{11} (-1)^k x^{2k}$

3. (2) Evaluate the following sums. (*Hint:* You may find it useful to review HW#4, question 6.)

(a) $\sum_{k=2}^{\infty} (0.3)^k$

Solution: This is an infinite geometric series with first term 0.09 and ratio 0.3, so the sum is $\frac{0.09}{1-0.3} = \frac{9}{70}$.

(b) $\sum_{k=1}^{\infty} k(1/4)^k$

Solution: The sum is $(1/4) + 2(1/4)^2 + 3(1/4)^3 + 4(1/4)^4 + \ldots$. Recall that

$$1 + 2x + 3x^2 + 4x^3 + \ldots = (1 + x + x^2 + x^3 + \ldots)^2 = \frac{1}{(1-x)^2}$$

Therefore, the given sum is equal to $\frac{1/4}{(1-1/4)^2} = \frac{1/4}{9/16} = \frac{4}{9}$.

(c) $\sum_{k=3}^{\infty} k^2(1/3)^k$

Solution: First, we evaluate the sum $\sum_{k=1}^{\infty} k^2(1/3)^{k-1} = 1+4(1/3)+9(1/3)^2+16(1/3)^3+\ldots$. Remember that

$$\begin{aligned} 1+4x+9x^2+16x^4+\ldots &= (1+3x+5x^2+7x^3+\ldots)(1+x+x^2+x^3+\ldots) \\ &= \left(\frac{2}{(1-x)^2}-\frac{1}{(1-x)}\right)\left(\frac{1}{1-x}\right) \\ &= \frac{2}{(1-x)^3}-\frac{1}{(1-x)^2} \end{aligned}$$

Hence,

$$\sum_{k=1}^{\infty} k^2(1/3)^k = \frac{2}{(2/3)^3}-\frac{1}{(2/3)^2} = \frac{27}{4}-\frac{9}{4} = \frac{9}{2}$$

Therefore, $\sum_{k=1}^{\infty} k^2(1/3)^k = \frac{3}{2}$, and so

$$\sum_{k=3}^{\infty} k^2(1/3)^k = \frac{3}{2}-(1/3)^1-4(1/3)^2 = \frac{3}{2}-\frac{1}{3}-\frac{4}{9} = \frac{13}{18}$$

4. (2) Kepler's third law of motion states that the ratio T^2/a^3 is constant, where a is a celestial body's average distance to the sun and T is its orbit period. The following table lists various celestial bodies in the solar system, along with their orbit period and their average distance to the Sun. Using a calculator, fill in the missing spaces. Then look these periods up online to check your answers.

Object name	Orbit period (y)	Average distance to Sun (km)
Earth	1.0	1.496×10^8
Mars	1.88	2.279×10^8
Mercury	0.241	5.794×10^7
Jupiter	11.862	7.781×10^8
Uranus	84.07	2.871×10^9
Pluto	247.94	5.904×10^9
1P/Halley	74.7	2.65×10^9

Solution: Calculations listed in order:

$$\left(\frac{2.279}{1.496}\right)^{3/2} \approx 1.88$$

$$(0.241)^{2/3} \cdot 1.496 \approx 0.5794$$

$$(11.862)^{2/3} \cdot 1.496 \approx 7.781$$

$$\left(\frac{28.71}{1.496}\right)^{3/2} \approx 84.07$$

$$(247.94)^{2/3} \cdot 1.496 \approx 59.04$$

$$(74.7)^{2/3} \cdot 1.496 \approx 26.5$$

5. **Solution:** Note: I have not included a typed solution to this problem at the moment.

(3) Consider the following experiment. Let N be some positive integer. Take the first N powers of 2, i.e. $2^0, 2^1, 2^2, \ldots, 2^{N-1}$. For each one, write down its leading digit. Then in this list of N numbers, count how many times each of the digits 1 through 9 appears, and write down these counts.

For example, when this is done for $N = 20$, the leading digits are

$$1, 2, 4, 8, 1, 3, 6, 1, 2, 5, 1, 2, 4, 8, 1, 3, 6, 1, 2, 5$$

The result is that we have six 1's, four 2's, two 3's, two 4's, two 5's, two 6'z, zero 7's, two 8's, and zero 9's. I.e., $(6, 4, 2, 2, 2, 2, 0, 2, 0)$.

(a) Do you expect the frequency of the different leading digits to eventually become equal as $N \to \infty$? That is, when N is very large, do you expect each digit to appear roughly $N/9$ times? Why or why not?

(b) Before you look at the next page, calculate the following values:

$$\log_{10}(2), \log_{10}(3/2), \log_{10}(4/3), \log_{10}(5/4)$$

You can use a calculator.

(c) The results of this experiment for various values of N are summarized in the table below:

Digit	$N = 100$	$N = 1000$	$N = 10,000$	$N = 100,000$	$N = 1,000,000$
1	30	301	3010	30103	301030
2	17	176	1761	17611	176093
3	13	125	1249	12492	124937
4	10	97	970	9692	96991
5	7	79	791	7919	79182
6	7	69	670	6695	66947
7	6	59	579	5797	57990
8	5	52	512	5116	51154
9	5	45	458	4575	45756

What pattern do you notice here?

(d) For each digit d, let $f(d, N)$ be the number of times the digit d appears as a leading digit among the numbers $\{1, 2, 4, \ldots, 2^{N-1}\}$. Then the fraction $\frac{f(d,N)}{N}$ converges to a particular value as $N \to \infty$. Can you guess what this value is? Can you explain why that's the value?