Omega HW #5 – Logarithms

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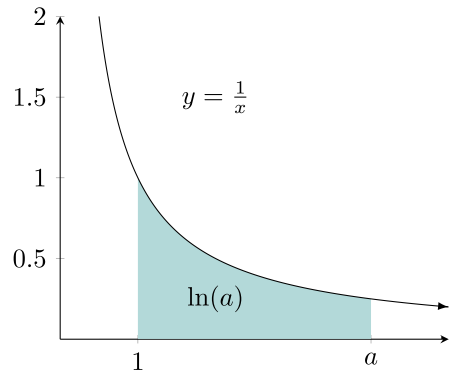
Logarithms

In class, we defined the *natural logarithm* of a, for any $a > 0$, as the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = a$.



We showed that $\ln(ab) = \ln(a) + \ln(b)$ for any a and b. We then defined the logarithm base-b by

$$\log_b(a) = \frac{\ln(a)}{\ln(b)}$$

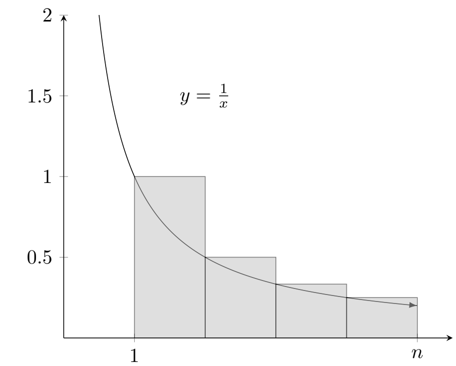
1. (1) Show the following facts, using the definition of the base-b logarithm above.
   (a) $\log_3(81) = 4$
   (b) $\log_a(b) \cdot \log_b(c) = \log_a(c)$
2. (1) Calculate the following without a calculator:
   (a) $\log_2(56) - \log_2(7)$
   (b) $5^{2\log_5(6)}$
   (c) $3\log_{10}(200) - \frac{1}{2}\log_{10}(64)$
3. (2) In this question, you will be asked to describe the relationship among various logarithmic functions.
   (a) Let $f(x) = \ln(x)$ and $g(x) = \ln(5x)$. What is the relationship between these two functions? Show that $g(x) = f(x) + C$ for some constant C. What is this constant?
   (b) Let $f(x) = \log_{1/8}(x)$ and $g(x) = \log_4(x)$. Show that $g(x) = Cf(x)$ for some constant C. What is this constant?

4. (2) Consider the **harmonic sequence**

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Let H_n denote the sum of the first n terms, namely $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. We will study the size of H_n as n grows.

(a) Show that $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} > \ln(n)$, using the following picture.



(b) Use a similar argument to show that $\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \ln(n)$.

(c) Using the previous two parts, show that $\ln(n+1) < H_n < \ln(n) + 1$.

(d) Does the infinite sum $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ converge to a finite value? Why or why not?

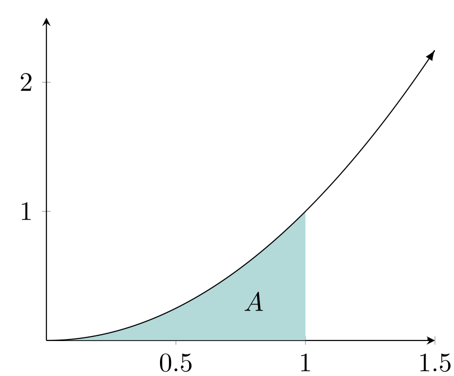
(e) **(Challenge!)** Use a similar style of argument to show the following two tighter inequalities:

$$\frac{1 + 1/2}{2} + \frac{1/2 + 1/3}{2} + \frac{1/3 + 1/4}{2} + \ldots + \frac{1/(n-1) + 1/n}{2} > \ln(n)$$

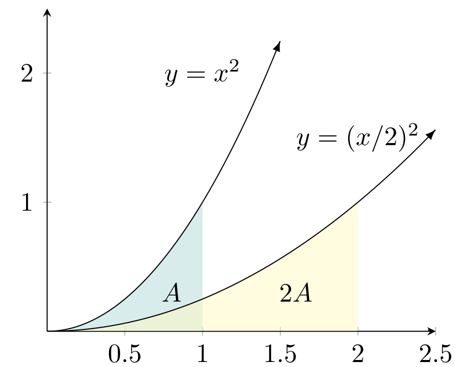
$$\frac{2}{1+2} + \frac{2}{2+3} + \frac{2}{3+4} + \ldots + \frac{2}{(n-1)+n} < \ln(n)$$

Quadrature of a Parabola

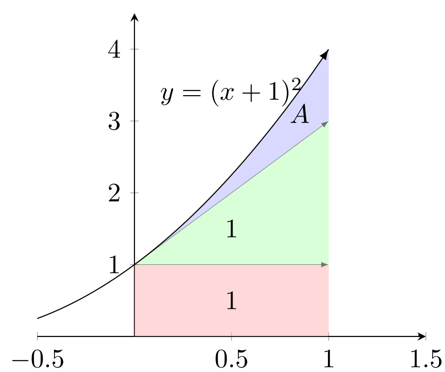
5. (2) In this problem, you'll be guided through the calculation of the area under the graph of the parabola $y = x^2$ from $x = 0$ to $x = 1$, labeled below as A.



(a) Argue geometrically that the area under the graph of $y = (x/2)^2$ from $x = 0$ to $x = 2$ equals $2A$. Then argue geometrically that the area under the graph of $y = x^2$ from $x = 0$ to $x = 2$ equals $8A$.



(b) Shift the graph to the left by 1 to get $y = (x + 1)^2 = x^2 + 2x + 1$. Show that the area under the graph of $y = 1$ from $x = 0$ to $x = 1$ is equal to 1, and also that the area under the graph of $y = 2x$ from $x = 0$ to $x = 1$ is also equal to 1. Argue that therefore, the area under the graph of $y = x^2 + 2x + 1$ from $x = 0$ to $x = 1$ is equal to $A + 1 + 1$.



(c) Combine the two calculations above to calculate the value of A.

6. (3) **Challenge:** Can you use the same technique to calculate the area under the graph of $y = x^3$ from $x = 0$ to $x = 1$? What about $y = x^4$. If you have done these correctly, you should spot a pattern: can you conjecture a formula for the area under the graph of x^n from $x = 0$ to $x = 1$?