Omega HW #4 – Geometric sequences and series

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A *geometric sequence* with ratio r is a sequence of numbers $a_0, a_1, a_2, \ldots$ where each term is r times the previous term, i.e. $a_n = ra_{n-1}$. If the first term is equal to c, then the n-th term is equal to cr^{n-1}. A *geometric series* is the sum of the terms in a geometric sequence. We proved the formula for a *finite* geometric series

$$c + cr + cr^2 + cr^3 + \ldots + cr^{n-1} = \frac{c(1-r^n)}{1-r}$$

We also proved that when $|r| < 1$, the *infinite* geometric series has sum

$$c + cr + cr^2 + cr^3 + \ldots = \frac{c}{1-r}$$

1. (1) Here you calculate a few geometric series sums.

   (a) Calculate the infinite geometric series $5 + \frac{5}{4} + \frac{5}{16} + \frac{5}{64} + \ldots$.
   Solution: The initial term is 5 and the ratio is 1/4, so the sum is $\frac{5}{1-\frac{1}{4}} = \frac{20}{3}$

   (b) Consider the geometric sequence whose zero-th and first terms are $a_0 = 2, a_1 = 3, \ldots$. Write a formula for the n-th term of the sequence. Then use this formula to calculate the sum of the first 5 terms.
   Solution: The n-th term is $2(\frac{3}{2})^n$. Thus, the sum of the first 5 terms is $\frac{2(1-(\frac{3}{2})^5)}{1-\frac{3}{2}} = \frac{2(-\frac{211}{32})}{-\frac{1}{2}} = \frac{211}{8}$.

2. (2) Show each of the following equalities by writing the decimal number on the left side as a geometric series, and then applying the formula for the sum of an infinite geometric series.

$$0.333333\ldots = 1/3$$

$$0.111111\ldots = 1/9$$

$$0.090909\ldots = 1/11$$

$$0.142857142857\ldots = 1/7$$

Solution:

$$\begin{aligned}
0.333333\ldots &= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots \\
&= \frac{3}{10}(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots) \\
&= \frac{3}{10}\left(\frac{10}{9}\right) = \frac{1}{3} \\
0.111111\ldots &= \frac{1}{10}(1 + \frac{1}{10} + \frac{1}{10^2} + \ldots) \\
&= \frac{1}{10}\left(\frac{10}{9}\right) = \frac{1}{9} \\
0.090909\ldots &= \frac{9}{100} + \frac{9}{10000} + \frac{9}{1000000} + \ldots \\
&= \frac{9}{100}(1 + \frac{1}{100} + \frac{1}{100^2} + \ldots) \\
&= \frac{9}{100}\left(\frac{100}{99}\right) = \frac{1}{11} \\
0.142857142857 &= \frac{142857}{1000000}(1 + \frac{1}{1000000} + \frac{1}{1000000^2} + \ldots) \\
&= \frac{142857}{1000000}\left(\frac{1000000}{999999}\right) = \frac{142857}{999999} = \frac{1}{7}
\end{aligned}$$

3. (1) Consider the *infinite* geometric series formula $c + cr + cr^2 + \ldots = \frac{c}{1-r}$. What would happen on both sides if $r = 1$ or $r = -1$? What if $|r| > 1$? Explain in a few sentences.

Solution: When $|r| \geq 1$, the sum $c + cr + cr^2 + \ldots$ does not converge to a finite value. For example,

- If $r = 1$, the sum is $c + c + c + \ldots$, which grows without bound. Correspondingly, the formula $\frac{c}{1-r}$ involves division by zero which is nonsensical.
- If $r = -1$, the sum $c - c + c - c\ldots$ alternates between 0 and c, never settling on a final value. The formula $\frac{c}{1-r}$ interestingly yields $\frac{c}{2}$, which is halfway between these two values.
- If $|r| > 1$, the terms cr^n grow in size without bound.

4. (2) The relationships $a_n = ra_{n-1}$ defining a geometric series is called a *linear recurrence relation* of order 1. In this question we study linear recurrence relations of order 2, i.e. $a_n = ra_{n-1} + sa_{n-2}$. We focus on $a_n = 5a_{n-1} - 6a_{n-2}$.

   (a) Show that the geometric sequence $a_0 = 2, a_1 = 6, a_2 = 18, a_3 = 54\ldots$ satisfies the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$.

   Solution: The general term of this sequence is $a_n = 2 \cdot 3^n$. So

$$\begin{aligned}
5a_{n-1} - 6a_{n-2} &= 5 \cdot 2 \cdot 3^{n-1} - 6 \cdot 2 \cdot 3^{n-2} \\
&= (15 - 6) \cdot 2 \cdot 3^{n-2} = 9 \cdot 2 \cdot 3^{n-2} = 2 \cdot 3^n = a_n
\end{aligned}$$

(b) Show that the geometric sequence $b_0 = 1, b_1 = 2, b_2 = 4, b_3 = 8\ldots$ also satisfies the same recurrence relation $b_n = 5b_{n-1} - 6b_{n-2}$.

Solution: The general term of this sequence is $b_n = 2^n$. So

$$\begin{aligned} 5b_{n-1} - 6b_{n-2} &= 5 \cdot 2^{n-1} - 6 \cdot 2^{n-2} \\ &= (10-6) \cdot 2^{n-2} = 4 \cdot 2^{n-2} = 2^n = b_n \end{aligned}$$

(c) Show that if we add these two geometric sequences term-wise to obtain the new sequence $c_0 = 3, c_1 = 8, c_2 = 22, c_3 = 62, \ldots$, then $c_n = 5c_{n-1} - 6c_{n-2}$.

Solution: We can deduce this directly from the first two parts, as

$$c_n = a_n + b_n = 5a_{n-1} - 6a_{n-2} + 5b_{n-1} - 6b_{n-2}$$

$$= 5(a_{n-1} + b_{n-1}) - 6(a_{n-2} - b_{n-2}) = 5c_{n-1} - 6c_{n-2}$$

(d) A sequence $a_0, a_1, a_2, a_3, \ldots$ is known to be a geometric sequence with some unknown ratio r, and is also known to satisfy the relationship $a_n = 5a_{n-1} - 6a_{n-2}$ for all $n \geq 2$. What could the possible values of r be? Why?

Solution: Write the n-th term of the sequence $a_n = a_0 r^n$. Then the recurrence relation becomes

$$a_0 r^n = 5a_0 r^{n-1} - 6a_0 r^{n-2} \implies r^n = 5r^{n-1} - 6r^{n-2} \implies r^2 = 5r - 6$$

Therefore, either $r = 2$ or $r = 3$.

(e) Can you write down another sequence $d_0, d_1, d_2, \ldots$ different from the one in part (c) which is *not* a geometric sequence, but still satisfies the recurrence $d_n = 5d_{n-1} - 6d_{n-2}$? Can you describe a method to make new such sequences?

Solution: Yes, another possible one could be given by $d_n = 2\cdot 3^n - 2^n$, i.e. $1, 4, 14, 46, \ldots$. More generally, any *linear combination* $d_n = a \cdot 3^n + b \cdot 2^n$, for fixed numbers a and b, will satisfy the given recurrence.

(f) A sequence $d_0, d_1, d_2, \ldots$ is defined by $d_0 = 0$, $d_1 = 1$, and $d_n = 5d_{n-1} - 6d_{n-2}$ for all $n \geq 2$. Find a general formula for the n-th term of this sequence.

Solution: Let $d_n = a \cdot 3^n + b \cdot 2^n$ for unknowns a and b. Then the first two terms of the sequence $d_0 = 0$ and $d_1 = 1$ imply

$$a \cdot 3^0 + b \cdot 2^0 = a + b = 0$$

$$a \cdot 3^1 + b \cdot 2^1 = 3a + 2b = 1$$

Subtracting twice the first equation from the second implies $a = 1$, and thus $b = -1$. Therefore, $d_n = 3^n - 2^n$.

6. (2) In this question, we look at infinite series whose terms are formed by multiplying a geometric sequence by another sequence. Let r be a number with $|r| < 1$.

(a) First we search for a formula for the infinite sum $1 + 2r + 3r^2 + 4r^3 + \ldots$. Show that

$$(1 + r + r^2 + r^3 + \ldots)^2 = 1 + 2r + 3r^2 + 4r^3 + \ldots$$

Solution: Expand the left side. For each n, the coefficient of r^n comes from $n+1$ possible products: $1 \cdot r^n, r \cdot r^{n-1}, r^2 \cdot r^{n-2}, \ldots, r^n \cdot 1$. So the coefficient of r^n is $n + 1$.

(b) Use this to calculate the value of the sum $1 + 2(1/3) + 3(1/3)^2 + 4(1/3)^3 + \ldots$.
Solution: $1 + 2(1/3) + 3(1/3)^2 + 4(1/3)^3 + \ldots = (1 + \frac{1}{3} + \frac{1}{3^2} + \ldots)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

(c) Next, we search for a formula for the infinite sum $1 + 4r + 9r^2 + 16r^3 + \ldots$. Show that

$$(1 + r + r^2 + r^3 + \ldots)(1 + 3r + 5r^2 + 7r^3 + \ldots) = 1 + 4r + 9r^2 + 16r^3 + \ldots$$

Solution: The coefficient for r^n when we expand the left side is $1+3+5+\ldots+(2n-1)$. Call this sum S. Then

$$1 + 3 + \ldots + (2n - 3) + (2n - 1) = S$$

$$(2n - 1) + (2n - 3) + \ldots + 3 + 1 = S$$

$$2n + 2n + \ldots + 2n + 2n = 2S$$

There are n terms in the sum on the left, so $2S = 2n^2 \implies S = n^2$.

(d) Use part (a) to find a formula for $1 + 3r + 5r^2 + \ldots$, and then use this to calculate

$$1 + 4(1/5) + 9(1/5)^2 + 16(1/5)^3 + 25(1/5)^4 + \ldots$$

Solution:

$$\begin{aligned} 1 + 3r + 5r^2 + \ldots &= (2 + 4r + 6r^2 + \ldots) - (1 + r + r^2 + \ldots) \\ &= 2(1 + r + r^2 + \ldots)^2 - (1 + r + r^2 + \ldots) \\ &= \frac{2}{(1-r)^2} - \frac{1}{1-r} \end{aligned}$$

Thus, $1 + 4r + 9r^2 + 16r^3 + \ldots = \frac{2}{(1-r)^3} - \frac{1}{(1-r)^2}$. So,

$$1 + 4(1/5) + 9(1/5)^2 + 16(1/5)^3 + \ldots = \frac{2}{(4/5)^3} - \frac{1}{(4/5)^2} = \frac{250}{64} - \frac{25}{16} = \frac{75}{32}$$

Bonus – Calculating square roots

7. (3) Remember the iterative method of calculating $\sqrt{a}$ discussed in Class #1: we begin with an initial guess x_0, and then we repeatedly apply the function $f(x) = \frac{1}{2}(x + \frac{a}{x})$ to get better approximations $x_1, x_2, x_3, \ldots$. In this question you'll analyze this method using the geometric series formula. For simplicity we'll set $a = 2$, so that the function is $f(x) = \frac{1}{2}(x + \frac{2}{x})$.

(a) Let $x = (1 + e)\sqrt{2}$ for some number e with $|e| < 1$. Show that $\frac{2}{x} = (1 - e + e^2 - \ldots)\sqrt{2}$.

Solution: Plug in the definition of x, and then use the formula for a geometric series with ratio $-e$.

$$\frac{2}{x} = \frac{2}{(1+e)\sqrt{2}} = \frac{\sqrt{2}}{1+e} = (1-e+e^2-\ldots)\sqrt{2}$$

(b) Use part (a) to show that $f(x) = (1 + \frac{e^2}{2(1+e)})\sqrt{2}$.

Solution: Since $x = (1+e)\sqrt{2}$, we have

$$\begin{aligned} x + \frac{2}{x} &= (2+e^2-e^3+e^4-\ldots)\sqrt{2} \\ &= (2+e^2(1-e+e^2-\ldots))\sqrt{2} \\ &= \left(2+\frac{e^2}{1+e}\right)\sqrt{2} \end{aligned}$$

Dividing by 2 on both sides gives $f(x) = (1 + \frac{e^2}{2(1+e)})\sqrt{2}$.

(c) Suppose that x is a fairly good approximation to $\sqrt{2}$ – e.g. $0.9\sqrt{2} < x < 1.1\sqrt{2}$. Show that $\sqrt{2} < f(x) < 1.01\sqrt{2}$.

Solution: Let $x = (1+e)\sqrt{2}$, and consider the formula $f(x) = (1 + \frac{e^2}{2(1+e)})\sqrt{2}$. We are given that $-0.1 < e < 0.1$. Then,

$$e^2 < 0.01 \qquad \text{and} \qquad 2(1+e) > 2(0.9) > 1$$

Thus, $\frac{e^2}{2(1+e)} < \frac{0.01}{1} = 0.01$. This implies that $f(x) < 1.01\sqrt{2}$. It is also clear that $f(x) > \sqrt{2}$ because $\frac{e^2}{2(1+e)} > 0$.

(d) Suppose that x is a very good approximation to $\sqrt{2}$ – e.g. $0.99\sqrt{2} < x < 1.01\sqrt{2}$. Show that $\sqrt{2} < f(x) < 1.0001\sqrt{2}$.

Solution: Again let $x = (1+e)\sqrt{2}$, and consider the formula $f(x) = (1 + \frac{e^2}{2(1+e)})\sqrt{2}$. We are given that $-0.01 < e < 0.01$. Then,

$$e^2 < 0.0001 \qquad \text{and} \qquad 2(1+e) > 2(0.99) > 1$$

Thus, $\frac{e^2}{2(1+e)} < \frac{0.0001}{1} = 0.0001$. This implies that $f(x) < 1.0001\sqrt{2}$. The same reasoning as before implies that $f(x) > \sqrt{2}$.

(e) Suppose that $x \approx \sqrt{2}$ is correct to d digits of accuracy, i.e., $(1-\frac{1}{10^d})\sqrt{2} < x < (1+\frac{1}{10^d})\sqrt{2}$. Show that $f(x) \approx \sqrt{2}$ is correct to about $2d$ digits of accuracy, i.e.

$$\sqrt{2} < f(x) < (1 + \frac{1}{10^{2d}})\sqrt{2}$$

Solution: Use the same reasoning: let $x = (1+e)\sqrt{2}$, and we are given that $-\frac{1}{10^d} < e < \frac{1}{10^d}$. Then,

$$e^2 < \frac{1}{10^{2d}} \qquad \text{and} \qquad 2(1+e) > 2(1-\frac{1}{10^d}) > 1$$

Thus, $\frac{e^2}{2(1+e)} < \frac{1}{10^{2d}}$. This implies that $f(x) < (1 + \frac{1}{10^{2d}})\sqrt{2}$. The same reasoning as before implies that $f(x) > \sqrt{2}$.

(f) Suppose we begin with the initial approximation $x_0 = 1.5$, which satisfies $x_0 < 1.1\sqrt{2}$ and apply the iterative method described. By the reasoning described above, at least how many digits of accuracy will x_4 have?

Solution: Applying part (e), we have

$$\begin{aligned} x_1 &< 1.01\sqrt{2} \\ x_2 &< 1.0001\sqrt{2} = \left(1 + \frac{1}{10^4}\right)\sqrt{2} \\ x_3 &< \left(1 + \frac{1}{10^8}\right)\sqrt{2} \\ x_4 &< \left(1 + \frac{1}{10^{16}}\right)\sqrt{2} \end{aligned}$$

Therefore, x_4 is accurate to *at least* 16 digits. (It turns out that $x_1 = \frac{17}{12}$ is correct to 2 digits after the decimal point, $x_2 = \frac{577}{408}$ is correct to 5 digits, $x_3 = \frac{665857}{470832}$ is correct to 11 digits, and $x_4 = \frac{886731088897}{627013566048}$ is correct to 23 digits.