Omega HW #3 – Conic sections, Part 2

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Parabolas, ellipses, and hyperbolas are the three types of conic sections. Last time you did some exercises involving parabolas ; this time, you will do some exercises involving ellipses and hyperbolas.

An ellipse or hyperbola is defined by a pair of points (called the *foci*) and a fixed length value L. An *ellipse* is the set of points such that the *sum* of the distances to the foci equals L. A *hyperbola* is the set of points such that the *difference* of the distances to the foci equals L.

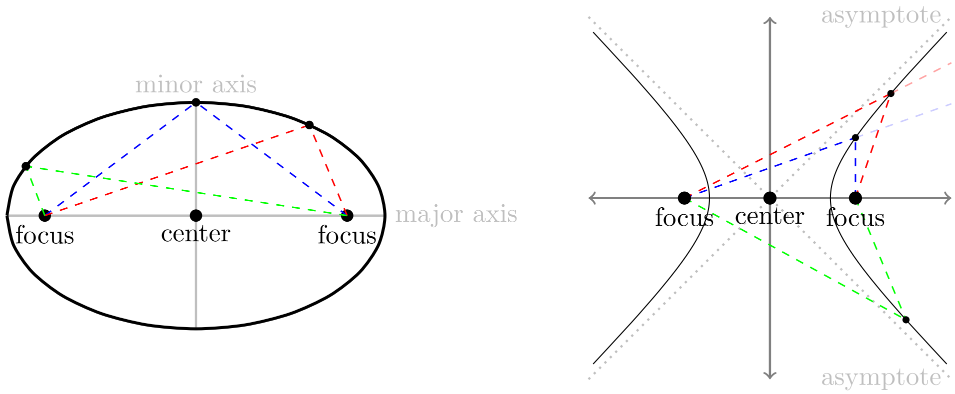
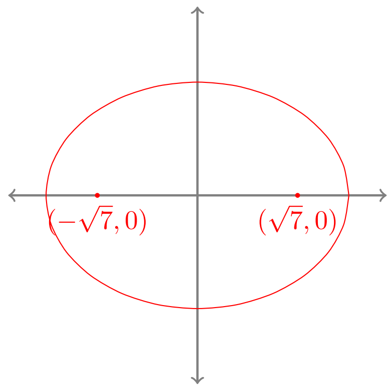


Figure 1: In an ellipse, the two segments connecting the foci to any point on the ellipse have sum independent of the point. Additionally, the two segments meet the ellipse at equal angles of incidence. In a hyperbola, the same properties hold, where the sum is replaced by the difference, and the segments are reflections of each other over the tangent line to the hyperbola.

Ellipses

1. (1) Consider the ellipse defined by the equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Sketch its graph, label the foci, and give their coordinates.

   Solution: The ellipse intersects the x-axis at $(\pm 4, 0)$ and the y-axis at $(0, \pm 3)$.

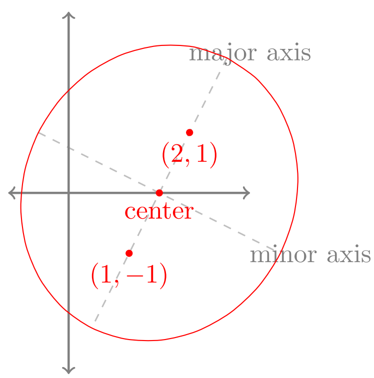


2. (2) Consider the ellipse with foci $(2, 1)$ and $(1, -1)$, and sum of distances 5.

   (a) Calculate the lengths of the major and minor axes.*

   (b) Show that the equation of this ellipse is $24x^2 - 4xy + 21y^2 - 72x + 6y - 71 = 0$.† *(Note: In the version of this homework distributed in class, this equation was written incorrectly)*

Solution:



The length of the major axis is equal to the sum of distances, which is 5. The distance between the foci is equal to $\sqrt{(2-1)^2 + (1+1)^2} = \sqrt{5}$, and so the length of the minor axis is $\sqrt{(\text{major axis})^2 - \sqrt{5}^2} = \sqrt{20} = 2\sqrt{5}$. To obtain the equation for the ellipse, we write down

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* *Hint:* The major axis can be immediately deduced from the sum of the distances. The minor axis can then be deduced by calculating the distance between the foci, and then drawing a certain right triangle to use the Pythagorean theorem.

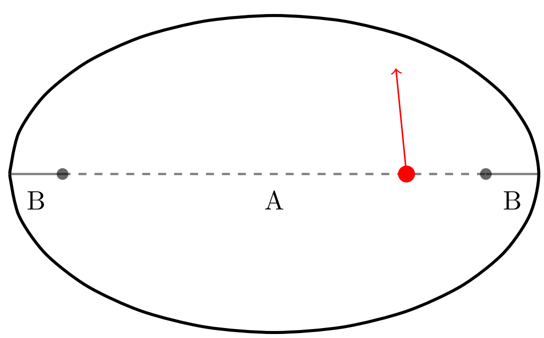
† *Hint:* Write down the defining condition of the ellipse as an equation, and then proceed by algebra to clear square roots.

the defining property and proceed by algebra.

$$\begin{aligned}
\sqrt{(x-2)^2+(y-1)^2}+\sqrt{(x-1)^2+(y+1)^2} &= 5 \\
\sqrt{(x-1)^2+(y+1)^2} &= 5-\sqrt{(x-2)^2+(y-1)^2} \\
(x-1)^2+(y+1)^2 &= 25-10\sqrt{(x-2)^2+(y-1)^2}+(x-2)^2+(y-1)^2 \\
(x-1)^2+(y+1)^2-(x-2)^2-(y-1)^2 &= 25-10\sqrt{(x-2)^2+(y-1)^2} \\
2x-3+4y &= 25-10\sqrt{(x-2)^2+(y-1)^2} \\
10\sqrt{(x-2)^2+(y-1)^2} &= 28-2x-4y \\
5\sqrt{(x-2)^2+(y-1)^2} &= 14-x-2y \\
25(x-2)^2+25(y-1)^2 &= (14-x-2y)^2 \\
25x^2-100x+25y^2-50y+125 &= 196+x^2+4y^2+4xy-28x-56y \\
24x^2-4xy+21y^2-72x+6y-71 &= 0
\end{aligned}$$

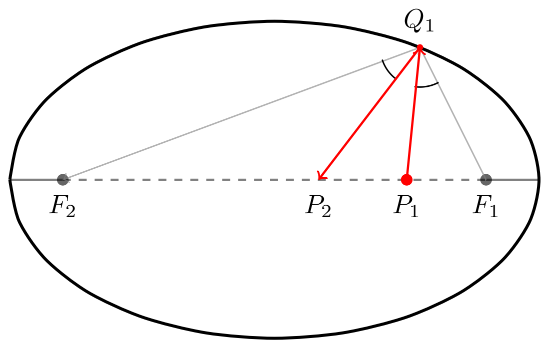
3. (3) In this question, you will reason about a trajectory traveling within an elliptical enclosure and how it bounces off the walls (e.g., a ray of light inside an elliptical mirror). In the notes, the property was given that a trajectory beginning at one focus and bouncing off the wall will reach the other trajectory: you can assume this.

   Divide the major axis of the ellipse into two regions as shown: A indicates the points between the foci, and B indicates the points not between the foci.



   (a) Argue that if a trajectory begins on the major axis in region A, then as time proceeds, it will only ever intersect the major axis in region A.

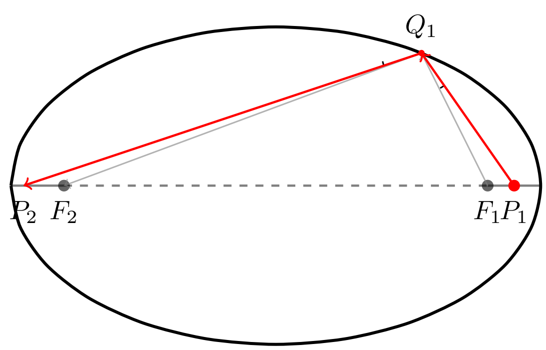
   Solution:



Let P_1 denote the starting point of the trajectory, let Q_1 denote the location where it first hits the ellipse, and let P_2 denote the location where it re-intersects the major axis. Consider the trajectory shown in gray, which begins at one focus F_1, bounces off the ellipse at P, and then travels to the other focus F_2. Since angles $\angle P_1Q_1F_1$ and $\angle F_2Q_1P_2$ are equal (due to the reflection property of the ellipse) and positive (because P_1 lies between F_1 and F_2), it follows that P_2 lies between F_1 and F_2. Thereafter, if we call the subsequent locations where the trajectory meets the ellipse $Q_2, Q_3, Q_4, \ldots$ and the subsequent intersections with the major axis $P_3, P_4, P_5, \ldots$, a similar argument yields that $P_3, P_4, P_5, \ldots$ all lie between F_1 and F_2.

(b) Argue that if a trajectory begins on the major axis in region(s) B, then as time proceeds, it will only ever intersect the major axis again in region(s) B.

Solution:

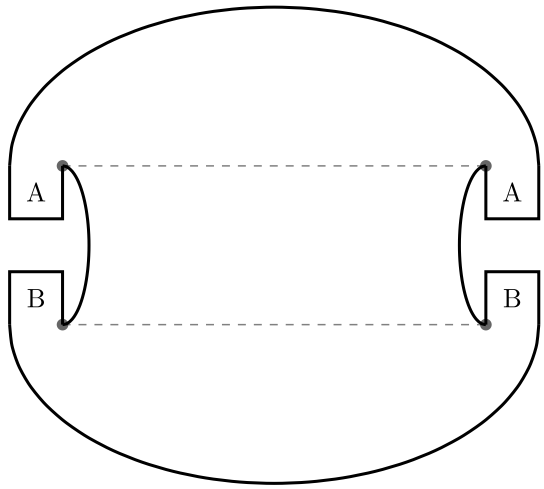


Apply a similar argument to part (a).

(c) The region below (sometimes called a *Penrose unilluminable room*) has the property that a single light source placed at any position within the shape cannot illuminate the entire region. More precisely, for any initial position within the room, if a light source is placed there then either the two square rooms labeled A will both remain dark, or the two square rooms labeled B will both remain dark. The shape is formed by two half-ellipses, along with several straight walls in the middle. Explain why this shape has this property.‡

‡*Hint:* The gray dashed lines divide the shape into three regions. Treat these as three separate cases.

Solution: Part (a) implies that light rays emanating from a source in the upper or middle third of the room can never reach the areas labeled (B), because they must cross the lower dashed line. Similarly, light rays emanating from a source in the lower or middle third of the room can never reach the areas labeled (A), because they must cross the upper dashed line. Therefore, there is no point in the room from which a single light source would illuminate the entire room.



Hyperbolas

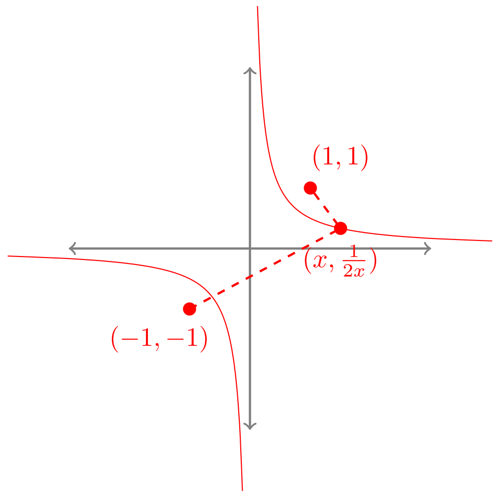
4. (1) Consider the graph of the function $y = \frac{1}{2x}$.

   (a) Sketch a graph of this function.

   (b) Choose any three different points $(x, \frac{1}{2x})$ on this graph. For each one, calculate its distance to both $(1, 1)$ and $(-1, -1)$, and show that the difference between these distances is equal to 2.

   (c) Prove that the graph of this equation is a hyperbola with foci $(1, 1)$ and $(-1, -1)$ and difference of distances 2. Do this by considering a generic point $(x, \frac{1}{2x})$ on the graph and computing its distances to both foci in terms of x, then showing their difference is 2.

   Solution:



First, we pick three example points on the hyperbola – $(1, \frac{1}{2}), (2, \frac{1}{4})$, and $(3, \frac{1}{6})$ – and for each compute the distance to $(1, 1)$ minus the distance to $(-1, -1)$.

$$\sqrt{(1-1)^2 + (\frac{1}{2}-1)^2} - \sqrt{(1+1)^2 + (\frac{1}{2}+1)^2} = \sqrt{\frac{1}{4}} - \sqrt{\frac{25}{4}} = \frac{1}{2} - \frac{5}{2} = -2$$

$$\sqrt{(2-1)^2 + (\frac{1}{4}-1)^2} - \sqrt{(2+1)^2 + (\frac{1}{4}+1)^2} = \sqrt{\frac{25}{16}} - \sqrt{\frac{169}{16}} = \frac{5}{4} - \frac{13}{4} = -2$$

$$\sqrt{(3-1)^2 + (\frac{1}{6}-1)^2} - \sqrt{(3+1)^2 + (\frac{1}{6}+1)^2} = \sqrt{\frac{169}{36}} - \sqrt{\frac{625}{36}} = \frac{13}{6} - \frac{25}{6} = -2$$

Now we consider a generic point $(x, \frac{1}{2x})$, where $x > 0$. We want to prove that

$$\sqrt{(x-1)^2 + \left(\frac{1}{2x}-1\right)^2} - \sqrt{(x+1)^2 + \left(\frac{1}{2x}+1\right)^2} \stackrel{?}{=} -2$$

$$\sqrt{(x-1)^2 + \left(\frac{1}{2x}-1\right)^2} \stackrel{?}{=} \sqrt{(x+1)^2 + \left(\frac{1}{2x}+1\right)^2} - 2$$

$$(x-1)^2 + \left(\frac{1}{2x}-1\right)^2 \stackrel{?}{=} (x+1)^2 + \left(\frac{1}{2x}+1\right)^2 - 4\sqrt{(x+1)^2 + \left(\frac{1}{2x}+1\right)^2} + 4$$

$$4\sqrt{(x+1)^2 + \left(\frac{1}{2x}+1\right)^2} \stackrel{?}{=} (x+1)^2 - (x-1)^2 + \left(\frac{1}{2x}+1\right)^2 - \left(\frac{1}{2x}-1\right)^2 + 4$$

$$4\sqrt{(x+1)^2 + \left(\frac{1}{2x}+1\right)^2} \stackrel{?}{=} 4x + \frac{2}{x} + 4$$

$$16(x+1)^2 + 16\left(\frac{1}{2x}+1\right)^2 \stackrel{?}{=} (4x + \frac{2}{x} + 4)^2$$

$$16x^2 + 32x + 16 + \frac{4}{x^2} + \frac{16}{x} + 16 \stackrel{?}{=} 16x^2 + \frac{4}{x^2} + 16 + 2(4x)(\frac{2}{x}) + 2(4x)(4) + 2(\frac{2}{x})(4)$$

Carefully simplifying all of the terms on the right side yields equality.

5. (2) Recall from class that the graph of the relationship $x^2 - y^2 = 1$ is a hyperbola with foci $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ and difference of distances 2.

   (a) What is the geometric relationship between this hyperbola and the hyperbola described in the previous problem?

   Solution: The hyperbola in the previous problem is a counterclockwise 45° rotation around the origin of this one, because that rotation sends the foci $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$ of this hyperbola to the foci $(1, 1)$ and $(-1, -1)$ of the other hyperbola.

   (b) Show, by algebraic expansion, that the equation

$$\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)^2 - \left(\frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}}x\right)^2 = 1$$

   is equivalent to the equation $xy = \frac{1}{2}$.

   Solution:

$$\begin{aligned}\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)^2 - \left(\frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}}x\right)^2 &= \left(\frac{x^2}{2} + xy + \frac{y^2}{2}\right) - \left(\frac{x^2}{2} - xy + \frac{y^2}{2}\right) \\ &= 2xy\end{aligned}$$

   Thus, the given equation is equivalent to $2xy = 1$, which is equivalent to $xy = \frac{1}{2}$.

   (c) Can you guess what graph would if you took another equation, e.g. the parabola $y = x^2$, and made the same substitution? That is, make a guess at the shape of the graph of the equation[§]

$$\frac{1}{\sqrt{2}}y - \frac{1}{\sqrt{2}}x = \left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)^2$$

   Solution: It turns out that this is a rotation of the parabola $y = x^2$ by 45° around the origin. This new parabola has focus $(-\frac{1}{4\sqrt{2}}, \frac{1}{4\sqrt{2}})$ and directrix $y = -x - \frac{1}{2\sqrt{2}}$.

[§] *Hint:* The substitution $(x, y) \mapsto (\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y, -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y)$ is a *rotation.* Later in the course, we will find a general formula for rotations.