Omega HW #2 – Conic sections (Solutions)

We explore some properties of parabolas using the tools of two-dimensional *coordinate geometry*. Remember that the distance between two points in the plane can be calculated using the Pythagorean theorem. For example, the distance between $(3,5)$ and $(-2,7)$ is equal to $\sqrt{(3-(-2))^2+(5-7)^2} = \sqrt{25+4} = \sqrt{29}$.

Parabolas

A *parabola* is the set of points in two-dimensional space that are of equal distance from a point (called the *focus*) and line (called the *directrix*). The point of the parabola that is closest to the directrix is called the *vertex*, and the *focal length* is the distance between the focus and the vertex.

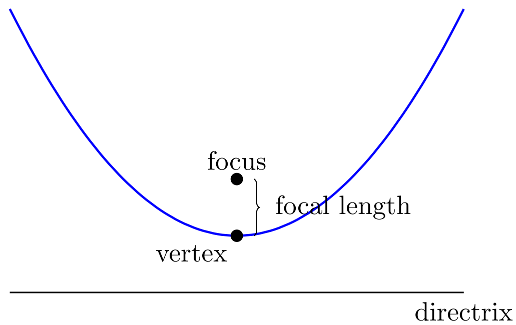


Figure 1: The parabola is shown in blue. All other components are labeled. The focal length is also equal to the distance from the vertex to the directrix.

For example, the parabola $y = x^2$ has its focus as $(0, \frac{1}{4})$, its directrix at $y = -\frac{1}{4}$, vertex at $(0,0)$, and focal length equal to $\frac{1}{4}$.

In general, the graph of a quadratic equation $y = ax^2 + bx + c$, for constants a, b, c with $a \neq 0$, is a parabola whose directrix is horizontal. You'll find formulas for the vertex, focus, and directrix of such a parabola in the questions below.

1. (1) Calculate the vertex, focus, and directrix of the parabola $y - 3 = x^2$.

   Solution: The graph of $y - 3 = x^2$ is obtained by shifting the graph of $y = x^2$ upwards by 3, so this means that the vertex lies at $(0,3)$, the focus at $(0, \frac{13}{4})$, and the directrix at $y = \frac{11}{4}$.

2. (1) Calculate the vertex, focus, and directrix of the parabola $y - 3 = (x+2)^2$.

   Solution: The graph of $y-3 = (x+2)^2$ is obtained by shifting the graph of $y-3 = x^2$ leftward by 2, so this means that the vertex lies at $(-2,3)$, the focus at $(-2, \frac{13}{4})$, and the directrix at $y = \frac{11}{4}$.

3. (1) Consider the parabola $y = 5x^2$. You'll calculate the focal length, which we'll call m.

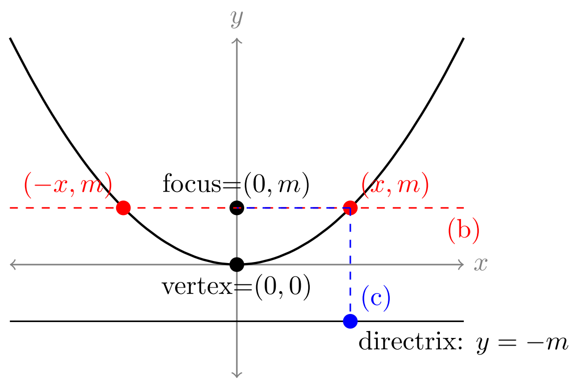
   (a) Sketch a graph of this parabola. Where is its vertex? Where are the focus and directrix, in terms of m?

   (b) Draw a horizontal line passing through the focus and mark the points where it intersects the parabola. What are the coordinates of these points, in terms of m?

   (c) Pick one of the two points you found and calculate its distance to both the focus and the directrix, in terms of m.

   (d) Set these two distances equal, and use this equation to solve for m.

Solution:



In the figure above, step (a) is drawn in black, step (b) in red, and step (c) in blue. The fact that the red points lie on the parabola implies that $m = 5x^2$, so $x = \sqrt{\frac{m}{5}}$. The two blue dashed line segments are, by the definition of a parabola, equal in length. We can write this as an equation, i.e. $x = 2m$. Using the fact that $x = \sqrt{\frac{m}{5}}$ yields $\sqrt{\frac{m}{5}} = 2m \implies \frac{m}{5} = 4m^2 \implies m = \frac{1}{20}$.

4. (2) Based on your reasoning in (3), give a general formula for the focal length m of the parabola $y = ax^2$, in terms of a. Then **prove** that every point on the parabola is equidistant from the focus $(0, m)$ and the directrix $y = -m$, as follows (similar to the proof covered in class):

   - Pick an arbitrary point (x, ax^2) on the parabola.
   - Calculate the distance from (x, ax^2) to both the focus and the directrix. You should get an algebraic expressions which depends on both a and x.
   - Show that they are identical.

Solution: A general formula for the focal length of $y = ax^2$ is given by $m = \frac{1}{4a}$, i.e. the focus is at $(0, \frac{1}{4a})$ and the directrix is at $y = -\frac{1}{4a}$. Pick any x and consider the point (x, ax^2). Its distance to the directrix equals $ax^2 + \frac{1}{4a}$, and its distance to the focus equals $\sqrt{x^2 + (ax^2 - \frac{1}{4a})^2}$. We would like these two to be equal, which we denote with $\stackrel{?}{=}$ as follows

$$\sqrt{x^2 + (ax^2 - \frac{1}{4a})^2} \stackrel{?}{=} ax^2 + \frac{1}{4a}$$

$$x^2 + (ax^2 - \frac{1}{4a})^2 \stackrel{?}{=} (ax^2 + \frac{1}{4a})^2$$
$$x^2 \stackrel{?}{=} (ax^2 + \frac{1}{4a})^2 - (ax^2 - \frac{1}{4a})^2$$

Simplifying the right hand side,

$$\begin{aligned}(ax^2 + \frac{1}{4a})^2 - (ax^2 - \frac{1}{4a})^2 &= 4(ax^2)\left(\frac{1}{4a}\right) \\ &= x^2\end{aligned}$$

so indeed the desired equality holds.

5. (2) Show that the vertex of the parabola $y = ax^2 + bx + c$ is at $(-\frac{b}{2a}, c - \frac{b}{4a})$.[1]

Solution: As suggested, we complete the square.

$$\begin{aligned}y &= ax^2 + bx + c \\ &= ax^2 + bx + \frac{b^2}{4a} + (c - \frac{b^2}{4a}) \\ &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})\end{aligned}$$

The square $(x + \frac{b}{2a})^2$ is always greater than or equal to 0. Therefore, if a is positive then y has a minimum value of $c - \frac{b^2}{4a}$ (and if a is negative, it is a maximum). In both cases, this extremal value is obtained at $x = -\frac{b}{2a}$, i.e. the point $(-\frac{b}{2a}, c - \frac{b^2}{4a})$ is the vertex of the parabola.

6. (2) Give the focus and directrix for the parabola $y = ax^2 + bx + c$.[2]

Solution: By completing the square, we know that $ax^2 + bx + c = a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})$. Thus, we can manipulate the given equation as follows.

$$\begin{aligned}y &= ax^2 + bx + c \\ y &= a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a}) \\ y - (c - \frac{b^2}{4a}) &= a(x + \frac{b}{2a})^2\end{aligned}$$

The graph of this equation is given by translating the graph of $y = ax^2$ so that its vertex lies at $(-\frac{b}{2a}, c - \frac{b^2}{4a})$. Since $y = ax^2$ has its focus at $(0, \frac{1}{4a})$ and its directrix at $y = -\frac{1}{4a}$, it follows that $y = ax^2 + bx + c$ has its focus at $(-\frac{b}{2a}, c - \frac{b^2}{4a} + \frac{1}{4a})$ and directrix at $y = c - \frac{b^2}{4a} - \frac{1}{4a}$.

[1] *Hint:* Rewrite the right-hand side of the equation using the following fact: $ax^2 + bx + \frac{b^2}{4a} = a(x + \frac{b}{2a})^2$.

[2] *Hint:* Relate this parabola to the parabola $y = ax^2$ by a translation. You calculated the focal length of the parabola $y = ax^2$ in (4). Now use the coordinates of the vertex as calculated in (5).

Bonus: Reflective properties of parabolas

A parabolic mirror has two miraculous reflective properties.[3]

- A ray emanating from the focus in any direction will bounce off of the parabola going in a trajectory perpendicular to the directrix. (When a ray hits a parabolic mirror, the angle of the incoming trajectory and the outgoing trajectory make the same angle with the line tangent to the parabola at the point of impact.)
- Two different rays sent out from the focus in different directions at the same speed, after bouncing off the mirror, end up at exactly the same distance away from the directrix.

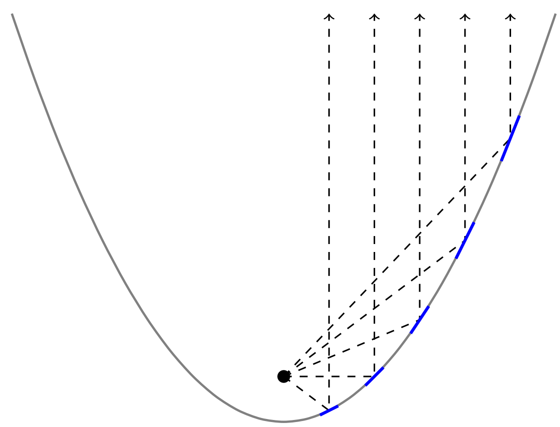


Figure 2: The five dashed trajectories shown all have the same total length. The tangent lines to the parabola at the points of incidence are shown in blue.

We'll sketch a proof of these two properties for the parabola $y = x^2$, whose focus is $(0, \frac{1}{4})$. You'll need to use two facts which we will encounter later in the course:

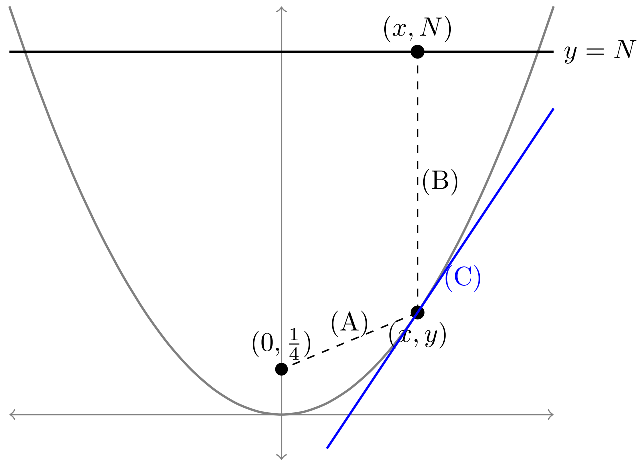
- The slope of the line tangent to the parabola $y = x^2$ at the point (a, b) has slope $2a$.
- If a line $\mathcal{L}$ has slope m, and the line $\mathcal{V}$ is a vertical line, then the reflection of $\mathcal{V}$ over $\mathcal{L}$ has slope $\frac{m^2-1}{2m}$.[4]

7. (2) Draw the parabola $y = x^2$ and the line $y = N$ for some large value of N. Pick a point (x, y) on the parabola with $y < N$, and calculate the distance from (x, y) to both the focus and to the line you drew. Show that the sum of these two distances is independent of the value of x which you picked.

   Solution:

[3]In optics, these two properties together imply that a parabolic mirror converts a spherical wave emanating from the focus, into a plane wave traveling orthogonal to the directrix, and vice versa. This is exactly why parabolic mirrors have the highest possible fidelity when used in reflecting telescopes, headlights, directional microphones, etc.

[4]This can be deduced from the addition formula for the $\tan(x)$ function.



The line segments from the chosen point to the focus and the line $y = N$ are labeled by (A) and (B), respectively. They have lengths $\sqrt{x^2 + (y - \frac{1}{4})^2}$ and $N - y$. Since $y = x^2$, we have

$$\begin{aligned}\sqrt{x^2 + (y - \frac{1}{4})^2} &= \sqrt{y + y^2 - \frac{y}{2} + \frac{1}{16}} \\ &= \sqrt{y^2 + \frac{y}{2} + \frac{1}{16}} \\ &= \sqrt{(y + \frac{1}{4})^2} \\ &= y + \frac{1}{4}\end{aligned}$$

Thus, the sum of the two lengths is $y + \frac{1}{4} + N - y = N + \frac{1}{4}$, which does not depend on the point (x, y) which was chosen.

8. (3) Calculate the slope of the line connecting (x, y) to the focus. Then use the two facts given above to prove that the angle between this line and the tangent line to the parabola at (x, y), is equal to the angle between the tangent line and the vertical line.

Solution: The tangent line (labeled (C)) at the point (x, y) has slope $2x$. The line (B) is vertical. Let (D) denote the reflection of line (B) over line (C). Then by the second given bullet point, the slope of (D) is $\frac{(2x)^2-1}{2(2x)} = \frac{4x^2-1}{4x} = \frac{x^2-\frac{1}{4}}{x}$. Meanwhile, the slope of the line connecting (x, y) to the focus (labeled (A)) is equal to $\frac{y-\frac{1}{4}}{x-0}$. Since $y = x^2$, this equals $\frac{x^2-\frac{1}{4}}{x}$. Thus, lines (A) and (D) are the same line, since they both pass through the point (x, y) and have the same slope. Thus, line (A) and line (B) are reflections of each other over the tangent line (C), as desired.