Omega HW #1 – Calculating functions (Solutions)

Square roots

In class, we talked about two methods of calculating $\sqrt{2}$. In the "decimal" method, we considered the decimal representation $\sqrt{2} = 1.d_1d_2d_3d_4\ldots$ and iteratively calculated d_1, d_2, d_3, and so on. In the "fractional" method, we begin with an initial approximation x (e.g., $x = 1$), and repeated applied the transformation $x \mapsto \frac{1}{2}\left(x + \frac{2}{x}\right)$ to get successively better approximations.

1. (1) Calculate $\sqrt{17}$ to at least 3 decimal places, using the decimal method.

   Solution:

$$
\begin{array}{rl}
 & 4\ .\ 1\ \ 2\ \ 3\ldots \\
\hline
\big) & 17\ .00\,00\,00\ldots \\
- & 16 \\
\hline
 & 1\ \ 00 \\
- & \ \ \ 80 \\
- & \ \ \ \ 1 \\
\hline
 & 19\,00 \\
- & 16\,40 \\
- & \ \ \ \ 4 \\
\hline
 & 2\,56\,00 \\
- & 2\,47\,20 \\
- & \ \ \ \ \ \ 9 \\
\hline
 & 8\,71 \\
 & \vdots
\end{array}
$$

2. (2) Calculate the first three fractional approximations of $\sqrt{17}$. *(Hint: You will need to change the transformation you use.)*

   Solution: We use the transformation $x \mapsto \frac{1}{2}(x+\frac{17}{x})$. If we begin with the initial approximation $x_0 = 1$, then our successive approximants are

$$x_1 = \frac{1}{2}(x_0 + \frac{17}{x_0}) = \frac{1}{2}(1 + \frac{17}{1}) = 9$$

$$x_2 = \frac{1}{2}(x_1 + \frac{17}{x_1}) = \frac{1}{2}(9 + \frac{17}{9}) = \frac{49}{9} \approx 5.4444\ldots$$

$$x_3 = \frac{1}{2}(x_2 + \frac{17}{x_2}) = \frac{1}{2}(\frac{49}{9} + \frac{17\cdot 9}{49}) = \frac{1889}{441} \approx 4.2834\ldots$$

   If we begin with a better initial guess, we will converge more quickly. For example, if $x_0 = 4$

then

$$x_1 = \frac{1}{2}(4 + \frac{17}{4}) = \frac{33}{8} = 4.125$$

$$x_2 = \frac{1}{2}(\frac{33}{8} + \frac{17 \cdot 8}{33}) = \frac{2177}{528} \approx 4.123106\dots$$

$$x_3 = \frac{1}{2}(\frac{2177}{528} + \frac{17 \cdot 528}{2177}) = \frac{9478657}{2298912} \approx 4.123106\dots$$

This final approximation is correct to 13 decimal places!

3. Let $f(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$.

   (a) (1) Calculate $f(\sqrt{2})$. **Solution:** $f(\sqrt{2}) = \frac{1}{2}(\sqrt{2} + \frac{2}{\sqrt{2}}) = \frac{1}{2}(\sqrt{2} + \sqrt{2}) = \boxed{\sqrt{2}}$.

   (b) (1) Draw a sketch of $y = f(x)$ for $1 \leq x \leq 2$. You can use a tool to help you, e.g.`https://www.desmos.com/calculator`. Indicate the following points in the graph:

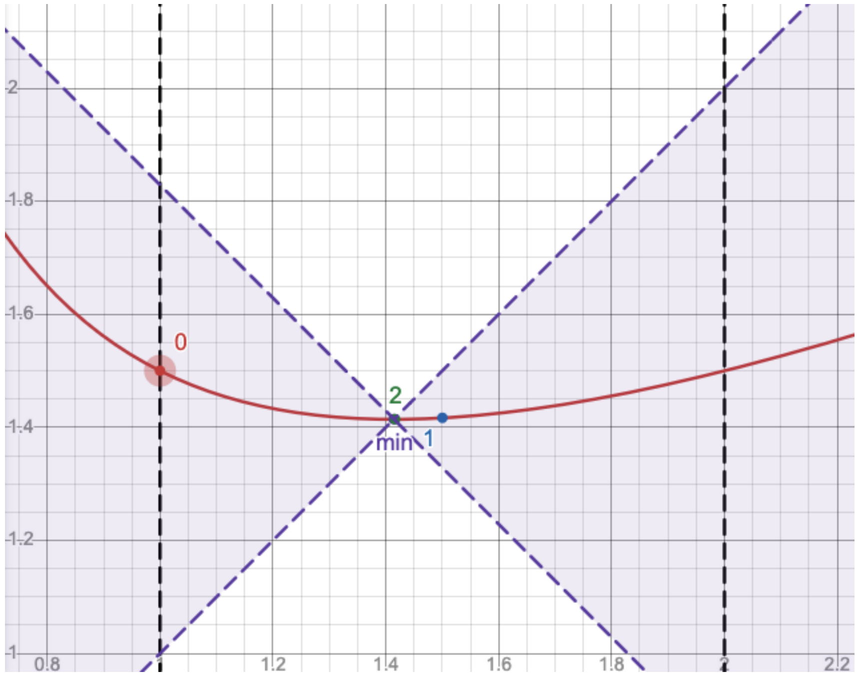
   - The minimum value of the function.
   - The points $(x, f(x))$ for the first three x values in the fractional method (beginning with $x = 1$).

     Solution: See the solution to the next part, where additional visualizations are marked.

   (c) (2) If x is a number in the interval $[1, 2]$, what can you say about the relative sizes of $|x - \sqrt{2}|$ and $|f(x) - \sqrt{2}|$, based on looking at this graph? Which is larger, and why?

   Solution: The graph below indicates the minimum value, as well as the first three iterates. The shaded regions indicate the regions where $|y - \sqrt{2}| < |x - \sqrt{2}|$. The graph of $y = f(x)$ is completely contained within these shaded regions, so this implies that for each iterate $x = x_0, x_1, x_2, \dots$, $f(x)$ will be closer to $\sqrt{2}$ than x is.[1]

[1] This graph was generated using Desmos – see `https://www.desmos.com/calculator/t298dirp3y`.



Functions and inverses

Given a function $f(x)$, its *inverse* $f^{-1}(x)$ (if it exists!) is a function such that

- $f^{-1}(f(x)) = x$ for all x in the domain of f, and
- $f(f^{-1}(x)) = x$ for all x in the domain of f^{-1}.

1 (1) Find the inverse of $f(x) = 4 - \frac{3}{4}x$.

Solution: The inverse must satisfy $x = 4 - \frac{3}{4}f^{-1}(x)$. Solving this equation gives $f^{-1}(x) = \frac{4}{3}(4 - x) = \frac{16}{3} - \frac{4}{3}x$.

2 (2) Let $f(x) = 3x^2 + 4x + 3$.

- Find a function $g(x)$ such that $f(g(x)) = x$. [2]

[2] *Hint:* Complete the square.

Solution 1: Using the hint, we complete the square.

$$\begin{aligned} f(x) &= 3\left(x^2+\frac{4}{3}x\right)+3 \\ &= 3\left(\left(x+\frac{2}{3}\right)^2-\frac{4}{9}\right)+3 \\ &= 3\left(x+\frac{2}{3}\right)^2-\frac{4}{3}+3 \\ &= 3\left(x+\frac{2}{3}\right)^2+\frac{5}{3} \end{aligned}$$

From here, we put $g(x)$ in place of x and x in place of $f(x)$ to get

$$\begin{aligned} x &= 3(g(x)+\frac{2}{3})^2+\frac{5}{3} \\ \frac{3x-5}{9} &= (g(x)+\frac{2}{3})^2 \\ -\frac{2}{3}\pm\sqrt{\frac{3x-5}{9}} &= g(x) \end{aligned}$$

We therefore get two possible solutions: $\boxed{g(x)=-\frac{2}{3}+\sqrt{\frac{3x-5}{9}}}$ and $\boxed{g(x)=-\frac{2}{3}-\sqrt{\frac{3x-5}{9}}}$

Solution 2: We directly solve $3g(x)^2+4g(x)+3=x$, which is equivalent to $3g(x)^2+4g(x)+(3-x)=0$, using the quadratic formula.

$$\begin{aligned} g(x) &= \frac{-8\pm\sqrt{16-4\cdot 3\cdot(3-x)}}{6} \\ &= \frac{-8\pm\sqrt{-20+12x}}{6} \\ &= \frac{-4\pm\sqrt{3x-5}}{3} \end{aligned}$$

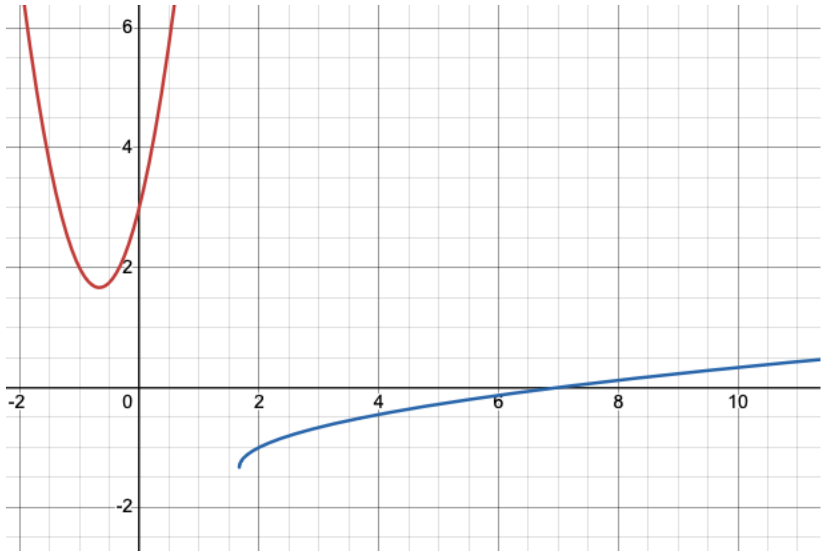
We therefore get two possible solutions for $g(x)$, similar to the previous solution.

- What are the domain and range of $g(x)$?

  Solution: For this question we consider the 'positive' branch, i.e. $g(x)=\frac{-4+\sqrt{3x-5}}{3}$. The function $g(x)$ is defined if and only if $3x-5\geq 0$, so the domain is $x\geq\frac{5}{3}$, alternatively written as $x\in[\frac{5}{3},\infty)$. The range is the set of values $g(x)$ can take on. Since the square root is always positive, we have $g(x)\geq-\frac{4}{3}$, so the range is $[-\frac{4}{3},\infty)$.

- Draw the graphs of both $y=f(x)$ and $y=g(x)$.

  Solution:



Bonus question: Cube roots and beyond

1. (2) Think about how you could adapt the decimal method to calculate *cube roots*. Use this method to calculate $\sqrt[3]{2}$ to 3 decimal places. I've included the first step below to get you started. [3]

   Solution:

$$
\begin{array}{rlr}
 & 1\,.\;\;2\;\;5\;\;9\ldots & \\
\hline
\big) & 2\,.\,000\,000\,000\ldots & \\
- & 1 & (=1^3) \\
\hline
 & 1\;\;000 & \\
- & 600 & (=3\cdot 10^2\cdot 2) \\
- & 120 & (=3\cdot 10\cdot 2^2) \\
- & 8 & (=2^3) \\
\hline
 & 272\,000 & \\
 & 216\,000 & (=3\cdot 120^2\cdot 5) \\
 & 9\,000 & (=3\cdot 120\cdot 5^2) \\
 & 125 & (=5^3) \\
\hline
 & 46\,875\,000 & \\
 & 42\,187\,500 & (=3\cdot 1250^2\cdot 9) \\
 & 303\,750 & (=3\cdot 1250\cdot 9^2) \\
 & 729 & (=9^3) \\
\hline
 & 4\,383\,021 & \\
 & \vdots &
\end{array}
$$

[3]Doing this by hand is computation-heavy, and requires multiplying 3-digit integers along the way. You can use a calculator to help you.

2. (3) Can you describe an algorithm to calculate $\sqrt[n]{x}$ for any whole number n? What about $x^{m/n}$ for any whole numbers m and n? (Remember that fractional exponents are defined by $x^{m/n} = (\sqrt[n]{x})^m$.)

Solution: For $\sqrt[n]{x}$, we could extend the same decimal method above to solve for successive digits, by using the coefficients in the expansion $(a+b)^n$. For example, suppose that A is an approximation to $\sqrt[n]{x}$ to the first $i-1$ digits after the decimal point, and $R = x - A^n$ denotes the remainder. We want to solve for the i-th digit d_i, using the equation

$$(A + \frac{d_i}{10^i})^n - A^n < R$$

and maximizing d_i. The coefficients in this expansion go $1, n, \frac{n(n-1)}{2}, \ldots$, and typically you'd find that it is enough to just use the first two terms, i.e. maximize d_i according to the constraint $n \cdot \frac{d_i}{10^i} \cdot A^{n-1} < R$.

To recover $x^{m/n}$, we just raise our result to the m-th power.

This method gets very computationally intensive as n gets larger, and so it is much more efficient to use a generalization of the fractional method (which we will get to once we discuss geometric series).