Omega Class #6 – Computing logarithms

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Recap: definition of logarithms

Last class we defined *logarithms*. The *base-b logarithm of a*, denoted $\log_b(a)$, is the unique number satisfying $b^{\log_b(a)} = a$. This definition only makes sense if b is a positive number not equal to 1, and $a > 0$. We talked about a few properties of logarithms, such as

$$\log_b(b) = 1$$

$$\log_b(ac) = \log_b(a) + \log_b(c)$$

$$\log_b(a^c) = c \cdot \log_b(a)$$

We then defined the *natural logarithm of a*, denoted $\ln(a)$, as the total area under the curve of $y = \frac{1}{x}$ from $x = 1$ to $x = a$. We showed that this function satisfies $\ln(ac) = \ln(a) + \ln(c)$, and so it is a logarithm. The base of this logarithm is called e: it's the unique number such that the area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = e$ is equal to 1. Next class we'll talk about how to calculate e exactly.

Today we'll talk about how to calculate logarithms.

The alternating harmonic series

So far, we have looked at a few sums of finite and infinite series. For example, two classes ago we found a formula for the sum of an infinite geometric series with ratio r such that $|r| < 1$:

$$c + cr + cr^2 + cr^3 + \ldots = \frac{c}{1-r}$$

This also held true for geometric series with negative ratio:

$$c - cr + cr^2 - cr^3 + \ldots = \frac{c}{1+r}$$

On your homework, you looked at the *harmonic sequence*, which is the sequence $1, 1/2, 1/3, 1/4, 1/5, \ldots$. In particular, one of the questions asked you to show that the infinite harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots$$

does not converge to a finite value, by showing that the partial sum $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \approx \ln(n)$.

Today we consider the *alternating harmonic series*:

$$\begin{aligned}
1.0 &= 1 \\
0.5 &= 1 - \frac{1}{2} \\
0.8333\ldots &= 1 - \frac{1}{2} + \frac{1}{3} \\
0.5833\ldots &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \\
0.7833\ldots &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \\
\vdots &= \vdots
\end{aligned}$$

Question: Does this infinite series reach a finite value? If so, what is it?

Here is a short snippet of code, written in the programming language Python, which calculates the n-th partial sum of the alternating harmonic series.

```
def alternating_harmonic_series(n):
    total = 0
    for i in range(1, n+1):
        total += ((-1)**(i-1)) / i
    return total
```

For comparison, here is a similar snippet of code which calculates the n-th partial sum of the harmonic series.

```
def alternating_harmonic_series(n):
    total = 0
    for i in range(1, n+1):
        total += ((-1)**(i-1)) / i
    return total
```

If you have access to a computer, this is a good chance to install Python and try running this computation. You should observe that the sum appears to converge to $0.6931\ldots$.

Sigma notation

This is a good opportunity to briefly introduce summation notation, also called *sigma* notation, after the Greek letter Σ (analogous to S). We use it for writing large sums. The part to the right of the Σ symbol tells us what terms we are summing in terms of an *index* (such as i), while the parts above and below the Σ tell us the range over which the index varies. This structure is very similar to the Python functions written above. Here are some examples, from which you can likely deduce the general pattern.

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{10} = \sum_{i=1}^{10} \frac{1}{i}$$

$$3+4+5+\ldots+13=\sum_{i=3}^{13} i$$

$$6+8+10+\ldots+24=\sum_{i=3}^{12} 2i$$

$$1+4+9+16+25+\ldots+121=\sum_{i=1}^{11} i^2$$

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\ldots-\frac{1}{30}=\sum_{i=1}^{30}\frac{(-1)^{i-1}}{i}$$

For infinite sums, we place the symbol ∞ on top of the sum. For example,

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i}$$

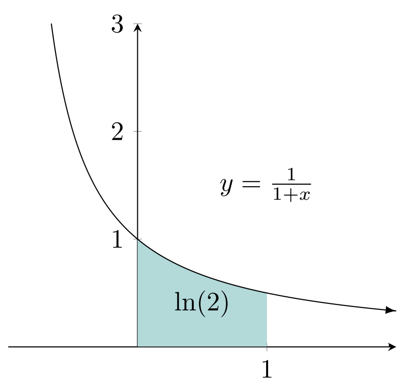
Calculating logarithms

It turns out that the alternating harmonic series sums to $\ln(2)$. I.e.,

$$\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i}=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots=\ln(2)$$

Here's a geometric proof of this fact.

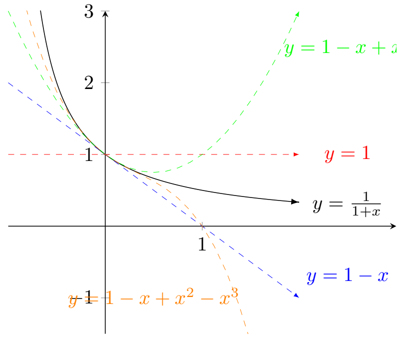
Proof: By definition, $\ln(2)$ is equal to the area under the graph of $y=\frac{1}{1+x}$ from $x=1$ to $x=2$. Let's shift that graph to the left by 1, so that $\ln(2)$ is the area under the graph of $y=\frac{1}{1+x}$ from $x=0$ to $x=1$.



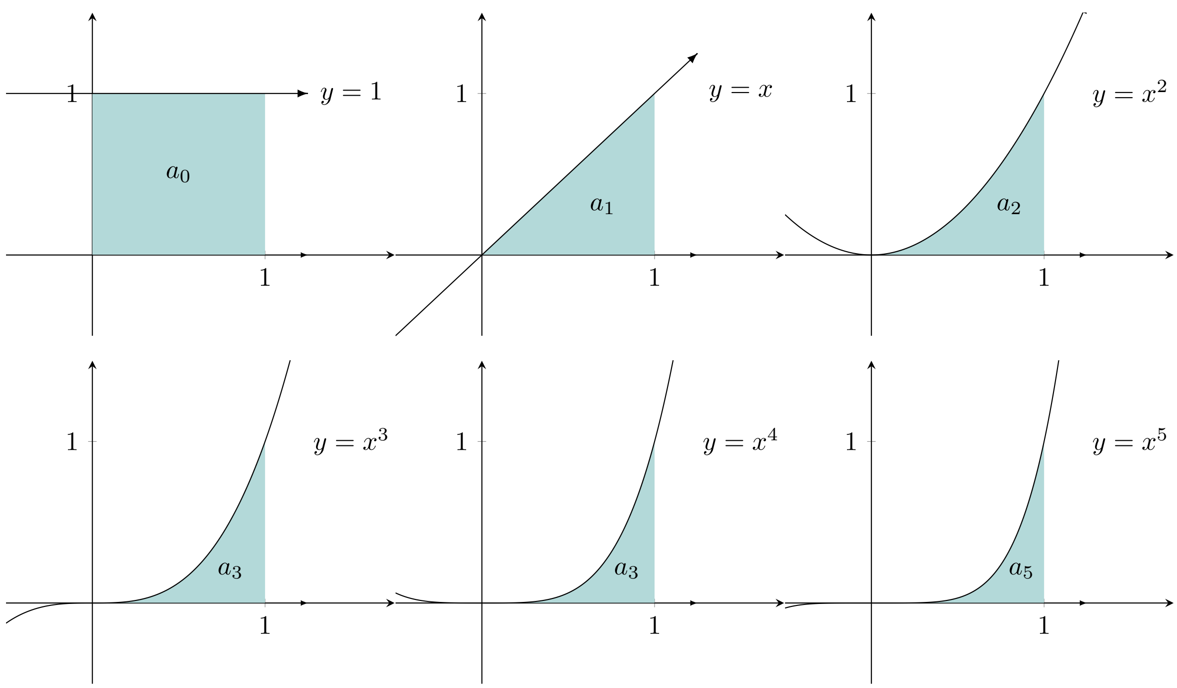
Next, recall from the formula for a geometric series that for every x such that $|x|<1$,

$$\frac{1}{1+x}=\sum_{i=0}^{\infty}(-x)^i=1-x+x^2-x^3+x^4-\ldots$$

What this means is that the *function* $\frac{1}{1+x}$ is better and better approximated by the partial sums of this function, i.e. $f_0(x) = 1$, $f_1(x) = 1 - x$, $f_2(x) = 1 - x + x^2$, $f_3(x) = 1 - x + x^2 - x^3$, etc, where $f_n(x)$ in general means the function $\sum_{i=0}^{n}(-x)^i$.



Let's write a_n to mean the area under the graph of $y = x^n$ from $x = 0$ to $x = 1$. Then by summation of areas, the area under the graph of $f_n(x) = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n$ from $x = 0$ to $x = 1$ is equal to $a_0 - a_1 + a_2 - a_3 + \ldots + (-1)^n a_n$.



It turns out that these values $a_0, a_1, a_2, \ldots$ can be calculated explicitly. For example, $a_0 = 1$ because it is the area of a square with sidelengths 1, while $a_1 = \frac{1}{2}$ because it is the area of a right triangle with legs of length 1 each. You showed in Question #5 of the last homework that $a_2 = \frac{1}{3}$. In general, it turns out that $a_n = \frac{1}{n+1}$, and this can be shown by methods similar to those in the homework assignment (but it is not that easy!).

This means that the area under the graph of $f_n(x)$ from $x = 0$ to $x = 1$ is equal to

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{(-1)^n}{n+1}$$

and the area under the graph of $f(x) = \frac{1}{1+x}$ from $x = 0$ to $x = 1$ is equal to

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Therefore, the alternating harmonic series sums to $\ln(2)$. ■

Calculating logarithms

A similar argument to the one given in the last section produces the following formula for computing the natural logarithm when $-1 < x < 1$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

This is our primary tool for calculating logarithms. This formula converges quickly when x is close to 0, because the individual terms $x, x^2/2, x^3/3, x^4/4, \ldots$ shrink faster than the terms of a geometric series with ratio x. Let's demonstrate a quick example:

Question: Calculate $\ln(1.5)$ to two decimal places of accuracy.

We plug in $x = 0.5$ to get

$$\ln(1.5) = 0.5 - \frac{0.5^2}{2} + \frac{0.5^3}{3} - \frac{0.5^4}{4} + \ldots$$

We want to get an answer to two decimal places of accuracy, i.e. to within 0.01. Notice that $\frac{0.5^5}{5} = \frac{1}{160}$ is smaller than 0.01. The terms after that one decrease faster than a geometric series of ratio 0.5, so their sum is less than $\frac{1}{160}$. So we can get to within the desired accuracy using the finite sum

$$0.5 - \frac{0.5^2}{2} + \frac{0.5^3}{3} - \frac{0.5^4}{4} + \frac{0.5^5}{5}6 = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{160} = \frac{480 - 120 + 40 - 15 + 6}{960} = \frac{391}{960} \approx 0.407$$