Omega Class #5 – Logarithms

Krishanu Sankar

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Definition of logarithms

Logarithms are a useful language for talking about relative sizes of objects.

Definition: The *logarithm base* b *of a positive number* a, is the unique number c such that $b^c = a$.

As a concrete example, let us consider the logarithm base 2. Consider the positive integer powers of 2, i.e. $1, 2, 4, 8, 16, 32, 64, \ldots$. Their base-2 logarithms are

$$\log_2(1) = 0, \quad \log_2(2) = 1, \quad \log_2(4) = 2, \quad \log_2(8) = 3, \ldots$$

In general, $\log_2(2^n) = n$. This also holds true for negative powers. For example,

$$\log_2\left(\frac{1}{2}\right) = -1, \quad \log_2\left(\frac{1}{4}\right) = -2, \quad \log_2\left(\frac{1}{8}\right) = -3, \ldots$$

The logarithm base 2 can be applied to numbers other than integer powers of 2. For example,

$$\log_2(\sqrt{2}) = \log_2(2^{1/2}) = \frac{1}{2}$$

$$\log_2\left(\frac{1}{4\sqrt{2}}\right) = 2^{-5/2} = -\frac{5}{2}$$

$$\log_2\left(8\sqrt[3]{2}\right) = 2^{3+1/3} = \frac{10}{3}$$

The logarithm base 2 also applies to positive numbers which are not expressible as a rational power of 2.

For example, consider $\log_2(10)$. This is supposed to be the number which you have to raise 2 to to get 10. You might expect this to be a little more than 3, since $2^3 = 8$. But it doesn't seem so simple to compute what this value should be.

It turns out that the answer, written in decimal form, is an infinite sequence with no simple pattern:

$$\log_2(10) = 3.3219281\ldots$$

When we want to write about a logarithm in a base other than 2, we would change the subscript. For example,

$$\log_3(9) = \log_3(3^2) = 2$$

$$\log_4 9(7) = \log_4 9(\sqrt{49}) = \frac{1}{2}$$

$$\log_{1/8}(2) = \log_{1/8}\left(\frac{1}{\sqrt[3]{1/8}}\right) = -\frac{1}{3}$$

Properties of logarithms

The base-b logarithmic function satisfies a few key properties:

1. For any two positive numbers m and n

$$\log_b(mn) = \log_b(m) + \log_b(n)$$

2. For any positive number m and any real number c,

$$\log_b(m^c) = c \cdot \log_b(m)$$

3. For any three positive numbers a, b, and c,

$$\log_a(b) \cdot \log_b(c) = \log_a(c)$$

These properties are direct consequences of the properties of exponents. The first property is equivalent to $b^x \cdot b^y = b^{x+y}$. The second and third properties follow from $a^{cx} = (a^x)^c$.

Question: Why is $\log_b(1) = 0$ for every base b?

Question: Why can't we take the logarithm of a negative numbers? For example, why can't we write $\log_2(-1)$? What about $\log_2(0)$?

Importantly, the third property tells us that the various logarithmic functions are all related to each other simply by scaling. For example,

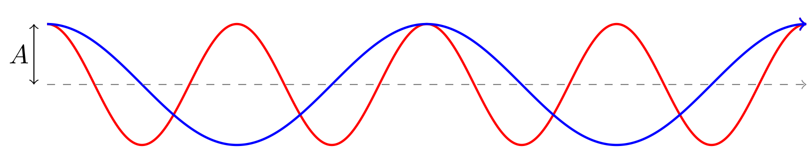
$$\log_3(x) = \log_2(x) \cdot \log_3(2)$$

The most commonly-used logarithms are $\log_{10}$ (in scientific and engineering applications) and $\log_2$ (in computer science and algorithms).

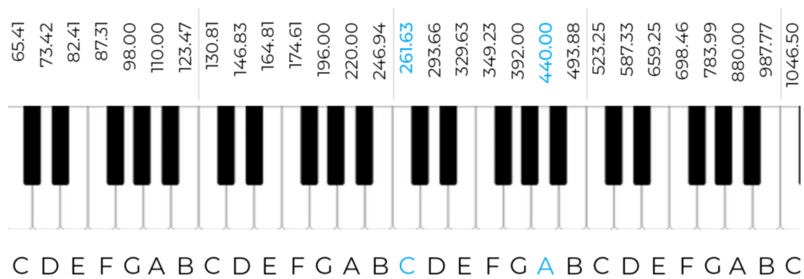
Uses of logarithmic scales

Many physical quantities appearing in nature are most naturally measured on a logarithmic scale.

For example, the pH of an acid or base measures the base-10 logarithm of the concentration of H^+ ions in solution. This is used because of the huge range of different concentration levels one encounters in nature.



Sound is an oscillating wave propagating through a medium, such as air or water. The number of oscillations per second is called the *frequency*, and the height or size of the wave is called the *amplitude*. For example, two waves are shown above with the same amplitude, but where the red wave has double the frequency.* Higher frequencies correspond to higher-pitched notes. The typical note scales we hear in music correspond to sound waves with frequencies whose logarithms are evenly spaced. For example, when one musical note has a frequency F and another has frequency $2F$, the two notes form an *octave*, and are even considered to be the same note. This is because of constructive interference between the sound waves which our eardrums pick up as resonance.

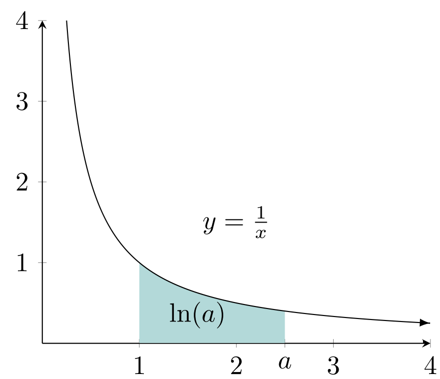


While the frequency of a sound wave is observable as pitch, the *amplitude* of a sound wave is observable as *volume*. Sound waves with a greater amplitude of oscillation are perceived as louder, and this is measured in *decibels* (dB), which is also a logarithmic scale. Every 20dB increase corresponds to multiplying the amplitude by a factor of 10. So for example, if a sound wave at 60dB has amplitude A, then a sound wave at 80dB has amplitude $10A$, at 100dB has amplitude $100A$, at 120dB has amplitude $1000A$, etc.

Electromagnetic radiation, which encompasses radio waves, visible light, microwaves, etc is also typically viewed on a logarithmic scale in terms of their frequency.

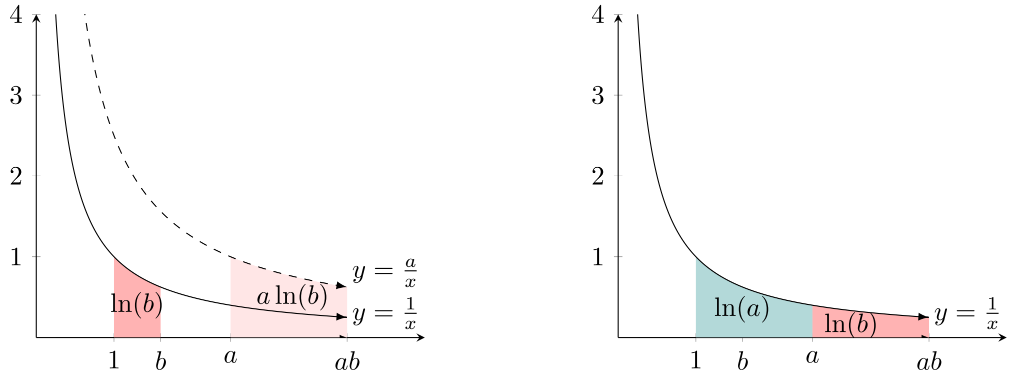
The natural logarithm

There is one logarithmic base b where the function $\log_b(x)$ can be visualized geometrically. Pick a positive number a. Consider the hyperbola described by function $y = 1/x$, and consider the area between this hyperbola and the x-axis between $x = 1$ and $x = a$. The measure of this area is called the *natural logarithm* of a, and is denoted $\ln(a)$.



*This image shows a *transverse* wave (e.g. light), but sound is embodied by a *longitudinal* wave. The mathematics works out the same for the two types of waves.

Let's take a moment to understand why this geometric construction has the multiplicative property, i.e. why is $\ln(ab) = \ln(a) + \ln(b)$. The key observation is to compare the graphs of $y = 1/x$ and $y = a/x$.



The second graph is a *factor-a horizontal stretch* of the first graph, which means that the area under the graph of $y = a/x$ from $x = a$ to $x = ab$ is equal to $a \cdot \ln(b)$. The second graph is a *factor-a vertical stretch* of the first graph, which means that the area under the graph of $y = 1/x$ from $x = a$ to $x = ab$ is equal to $\frac{a \cdot \ln(b)}{a} = \ln(b)$. From here, we deduce that the area under the graph of $y = 1/x$ from $x = 1$ to $x = ab$ can be broken into two parts of respective sizes $\ln(a)$ and $\ln(b)$, which implies that $\ln(ab) = \ln(a) + \ln(b)$.

There is a number, called e, such that $\ln(x) = \log_e(x)$. This number is approximately equal to $e \approx 2.71828\ldots$.