Omega Class #4 – Geometric Series

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Today's class is a bit more theoretical, and introduces both geometric sequences and recurrence relations. Both of these appear in many places in mathematics. Additionally, we'll calculate a formula for the sum of terms of a geometric sequence – called a *geometric series* – which is a tool used heavily throughout mathematics, including within calculus.

It's worth mentioning here that some of the objects you'll see in the next few classes have continuous analogues you may see in future calculus classes:

Discrete version	Continuous version (calculus)
Arithmetic sequence	Linear function
Geometric sequence	Exponential function
Recurrence relation	Differential equation
Summation	Integral
Generating function*	Laplace transform

Finite geometric series

Let's start with a concrete example.

Question: Find a simple formula for the sum $1 + 2 + 4 + \ldots + 2^{n-1}$.

Listing out some examples:

$$\begin{aligned}
1+2 &= 3 \\
1+2+4 &= 7 \\
1+2+4+8 &= 15 \\
1+2+4+8+15 &= 31 \\
1+2+4+8+15+32 &= 63
\end{aligned}$$

We could guess the formula $1 + 2 + 4 + \ldots + 2^{n-1} = 2^n - 1$. We can prove this formula with the following algebraic trick. Let S_n denote the sum $1+2+4+\ldots+2^{n-1}$. Then $2S_n = 2+4+8+\ldots+2^n$. When we subtract S_n, we observe a cancellation of terms:

$$\begin{array}{cccccccccccccc}
 & (& & 2 & + & 4 & + & \ldots & + & 2^{n-1} & + & 2^n & = & 2S_n) \\
- & (1 & + & 2 & + & 4 & + & \ldots & + & 2^{n-1} & & & = & S_n) \\
\hline
 & -1 & & & & & & & & & & 2^n & = & S_n
\end{array}$$

The sequence $1, 2, 4, 8, \ldots$ is an example of a *geometric sequence*, where each term is equal to the previous term times a fixed factor. The factor, which is 2 in this case, is called the *ratio*. The sum of the first n terms of this sequence is called a *finite geometric series*.

If we change the ratio to some arbitrary value r, so that our geometric sequence is $1, r, r^2, r^3, \ldots$, then the same trick can be used to obtain a cancellation of terms, but multiplying by r instead of 2.

$$\begin{array}{ccccccccccccc} & & r & + & r^2 & + & \dots & + & r^{n-1} & + & r^n & = & rS_n \\ 1 & + & r & + & r^2 & + & \dots & + & r^{n-1} & & & = & S_n \\ \hline -1 & & & & & & & & & & r^n & = & (r-1)S_n \end{array}$$

We therefore obtain $S_n = \frac{r^n - 1}{r-1}$. If the first term of the sequence is no longer 1, but is some other initial value c, then the sum of the first n terms is given by

$$c + cr + cr^2 + \dots cr^{n-1} = \frac{c(r^n - 1)}{r - 1}$$

This formula works for all values of r except for $r = 1$, including fractional and negative values. For example,

$$1 + 3 + 9 + \dots + 3^{n-1} = \frac{3^n - 1}{2}$$

$$1 + \frac{1}{5} + \frac{1}{25} + \dots + \frac{1}{5^{n-1}} = \frac{\frac{1}{5^n} - 1}{\frac{1}{5} - 1} = \frac{5}{4}\left(1 - \frac{1}{5^n}\right)$$

$$1 - \frac{1}{2} + \frac{1}{4} + \dots + \frac{(-1)^{n-1}}{2^{n-1}} = \frac{\left(-\frac{1}{2}\right)^n - 1}{-\frac{1}{2} - 1} = \frac{2}{3}\left(1 - \left(-\frac{1}{2}\right)^n\right)$$

Of course, when $r = 1$, it's easy to see that the sum is $c \cdot n$.†

Infinite geometric series

Let's consider the case of a value of r less than 1, e.g. $r = \frac{1}{2}$. What happens when we try to sum *all* of the terms of the geometric sequence, i.e. $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$? This is an *infinite* geometric series. Its value can be calculated by the same trick:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \dots = \frac{1}{2}S$$

$$\implies 1 = S - \frac{1}{2}S = \frac{1}{2}S$$

$$\implies S = 2$$

Like before, we can replace the ratio by an arbitrary number r and use the same trick to calculate the infinite sum. Note that this infinite sum only makes sense when $-1 < r < 1$, because the terms must be shrinking in size in order for the infinite sum to converge.‡

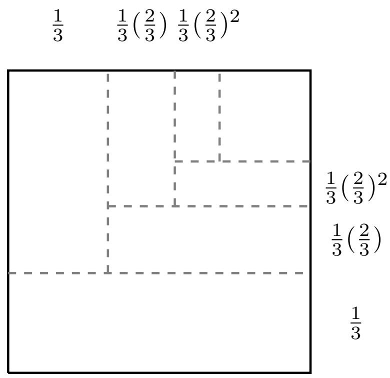
In general, if the first term of the geometric sequence is c and the ratio is r for $|r| < 1$, then the infinite sum is

$$c + cr + cr^2 + cr^3 + \dots = \frac{c(-1)}{r - 1} = \frac{c}{1 - r}$$

This formula has a nice visualization, shown below for $r = 2/3$.

† *Question:* Why does the proof we used break down when $r = 1$?

‡ It is possible to write down infinite sequences whose terms are shrinking in size, but where the infinite sum still does not converge, for example $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$.



The terms of the geometric series

$$(1-r) + r(1-r) + r^2(1-r) + r^3(1-r) + \ldots = 1$$

correspond to the areas of the rectangles, which fill a square of sidelength 1.

Recurrence relations

We're going to introduce some slightly more general notation to describe geometric series. Let

$$a_0, a_1, a_2, a_3, \ldots$$

denote an infinite sequence of numbers. Then this is a geometric sequence with ratio r if and only if

$$a_n = r a_{n-1} \qquad \text{for all } n \geq 1$$

The equation relating a_n and a_{n-1} is an example of a *recurrence relation*, or simply a *recurrence*. It describes how to get each term of the sequence from the previous one. This information, plus the first term a_0, fully determines the geometric sequence.

We can use recurrence relations to describe other familiar sequences. For example, an *arithmetic sequence*, which is one where the difference between consecutive terms is constant (for example, $1, 4, 7, 10, 13, \ldots$, is described by the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2}$$

This relation defines each term of the sequence in terms of the previous *two* terms. If you know the first two terms of the sequence, you can sequentially construct all of the rest.

There's a famous sequence, often known as the Fibonacci sequence[§] defined by $F_0 = 0$, $F_1 = 1$, and the recurrence relation $F_n = F_{n-1} + F_{n-2}$. The sequence goes as follows:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

A bit of numerical computation suggests that the successive ratios $\frac{F_n}{F_{n-1}}$ stabilize to a value of roughly

[§]This name comes from the Italian mathematician Leonardo Bonacci from the 1100s, but has been known in many other cultures for ages before that. This sequence has a great deal of historical significance, and is worth reading about.

1.618…, known as the *golden ratio*, often written as ϕ. More precisely, this number is $\phi = \frac{1+\sqrt{5}}{2}$. That is, the Fibonacci sequence roughly approximates a geometric sequence with ratio ϕ.

Question: Can you find a general formula for the n-th Fibonacci number, F_n?

We'll answer this question in the next class.