Omega Class #3 – Conic Sections, Part 2

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Last class we introduced conic sections and looked at parabolas. In particular, we proved that for any fixed constant m, the graph of the equation $y = mx^2$ is the set of points equidistance from the point $(0, \frac{1}{4m})$ and the line $y = -\frac{1}{4m}$. In these notes, we'll look at ellipses and hyperbolas.

Ellipses from stretching the unit circle

The equation $x^2 + y^2 = 1$ describes a circle centered at $(0, 0)$ with radius 1. This is because the distance from (x, y) to $(0, 0)$ is equal to $\sqrt{x^2 + y^2}$, and setting this equal to 1 and squaring yields the equation above.

By the same reasoning, the equation to describe a circle centered at $(0, 0)$ with radius r, for any fixed $r > 0$, is $x^2 + y^2 = r^2$. Another way to see this is to take the unit circle and stretch it by a factor of r in both the x-direction and y-direction, by applying the substitutions $x \mapsto \frac{x}{r}$ and $y \mapsto \frac{y}{r}$, yielding the equation $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$, which is equivalent to $x^2 + y^2 = r^2$.

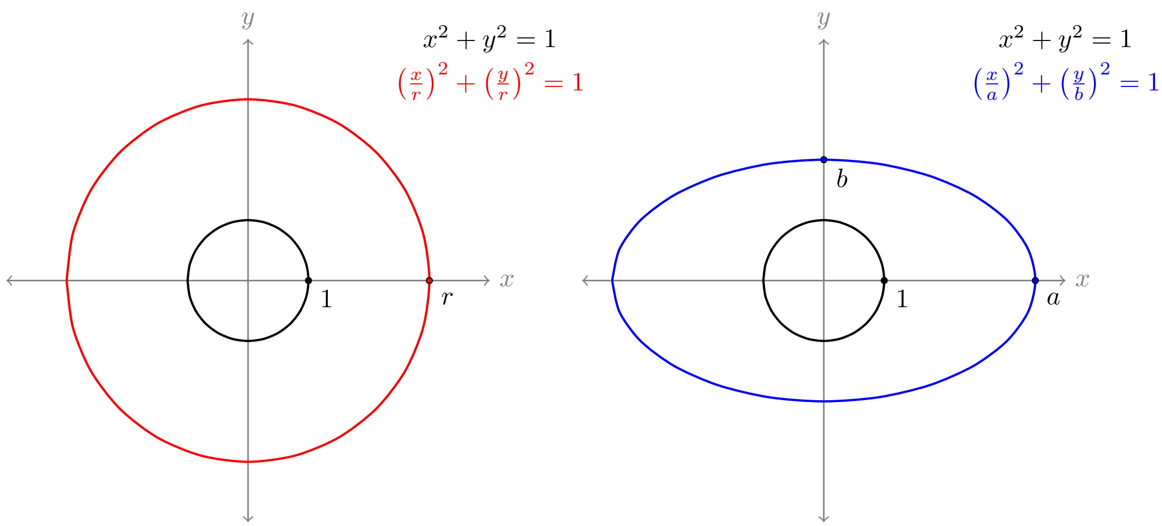

Figure 1: Stretching a unit circle by different factors in the x and y directions produces an ellipse. When the two factors are equal, this ellipse is a circle.

If you wanted to stretch the unit circle by a *different* ratio in each direction, so that the horizontal radius is a and the vertical radius is b, for some different numbers a and b, you would take the equation $x^2 + y^2 = 1$ and substitute $x \mapsto \frac{x}{a}$ and $y \mapsto \frac{y}{b}$ to get the equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. This is an *ellipse*. The shortest radius is called the *minor radius* and the longest is called the *major radius*. When the two have the same length, the ellipse is a circle.

Finding the foci of an ellipse

Suppose you are given two points P_1 and P_2 with distance c between them, and another number d such that $d > c$. Consider the set of points Q such that the sum of the lengths of P_1Q and P_2Q is equal to d. One way you can draw this is by placing thumbtacks at P_1 and P_2, placing a loop of string with length $c + d$ around the two thumbtacks and pulling it taut with a pencil at the third point, and then moving the pencil while keeping the string taut.

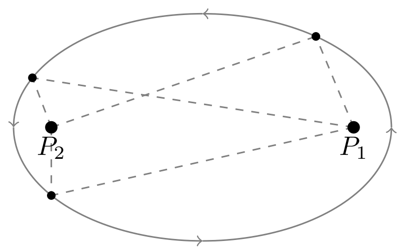

Figure 2: Drawing an ellipse as the set of points whose summed distances to P_1 and P_2 is fixed.

Surprisingly, it turns out that this procedure makes an ellipse. Let's pause for a moment to think about why this is surprising. It's that the first procedure of stretching a circle produces a kind of oblong, round shape, and you can convince yourself that this second procedure of tracing a point whose summed-distances to two fixed points also produces some kind of round shape which is longer along the line passing through the foci. But it's not obvious at all these these should be the *same* shape!

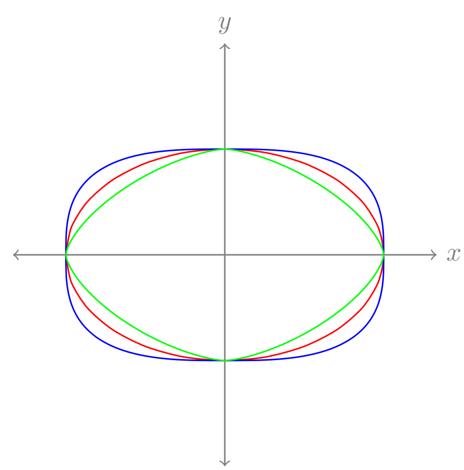

Figure 3: It's possible to have different curved shapes which intersect the x-axis and y-axis at the same points.

Why are these the same? We will show that these two characterizations are equivalent in a moment, but first let's assume we know this. Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and suppose its foci are at $(c, 0)$ and $(-c, 0)$ for some number c. We will calculate c and the summed distance in terms of a and b.

- The point $(a, 0)$ lies on the ellipse, and the sum of the distances from this point to the two foci equals $(a - c) + (a + c) = 2a$. Therefore, the summed distance is equal to $2a$.

- The point $(0, b)$ lies on the ellipse, and the sum of the distances from this point to the two foci equals $\sqrt{b^2+c^2} + \sqrt{b^2+c^2} = 2\sqrt{b^2+c^2}$. Therefore, $\sqrt{b^2+c^2} = a$, which implies that $c = \sqrt{a^2-b^2}$.

Now we'll prove the following proposition.

Proposition: Let $a > b$. The shape described by the equation $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is the set of points whose summed-distance to the two foci $(\sqrt{a^2-b^2}, 0)$ and $(-\sqrt{a^2-b^2}, 0)$ is equal to $2a$.

Proof: For simplicity of notation, let c denote the quantity $\sqrt{a^2-b^2}$. Let (x, y) be a point on the graph of $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$. Call the sum of its distances to the two foci D. Then

$$D = \sqrt{(x-c)^2+y^2} + \sqrt{(x+c)^2+y^2}$$

We will perform a sequence of algebraic steps to clear the square roots and try to get a better expression for D.

$$\begin{aligned}
D - \sqrt{(x-c)^2+y^2} &= \sqrt{(x+c)^2+y^2} \\
(D - \sqrt{(x-c)^2+y^2})^2 &= (\sqrt{(x+c)^2+y^2})^2 \\
D^2 - 2D\sqrt{(x-c)^2+y^2} + (x-c)^2 + y^2 &= (x+c)^2 + y^2 \\
D^2 - 2D\sqrt{(x-c)^2+y^2} &= (x+c)^2 - (x-c)^2 \\
D^2 - 2D\sqrt{(x-c)^2+y^2} &= 4xc \\
D^2 - 4xc &= 2D\sqrt{(x-c)^2+y^2} \\
(D^2 - 4xc)^2 &= (2D\sqrt{(x-c)^2+y^2})^2 \\
D^4 - 8D^2xc + 16x^2c^2 &= 4D^2((x-c)^2+y^2) \\
D^4 - 8D^2xc + 16x^2c^2 &= 4D^2x^2 - 8D^2xc + 4D^2c^2 + 4D^2y^2 \\
D^4 - D^2(4x^2+4c^2+4y^2) + 16x^2c^2 &= 0
\end{aligned}$$

Substituting $c^2 = a^2 - b^2$ (which we know), and the guess $D = 2a$ yields

$$16a^4 - 4a^2(4x^2 + 4a^2 - 4b^2 + 4y^2) + 16x^2(a^2 - b^2) = 0$$

$$a^4 - a^2x^2 - a^4 + a^2b^2 - a^2y^2 + x^2a^2 - x^2b^2 = 0$$

$$a^2b^2 - a^2y^2 - x^2b^2 = 0$$

$$1 - \frac{y^2}{b^2} - \frac{x^2}{a^2} = 0$$

which is the equation defining the ellipse. Hence, $D = 2a$ holds, for any values of x and y. ■

Lastly, ellipses satisfy two other properties which are highly analogous to parabolas.[1]

- Suppose that the summed-distance is equal to d, and draw the circle centered at one focus with radius d. Then every point on the ellipse is equidistant from the circle and the other focus.
- Pick any point P on the ellipse, and draw the tangent line at P. Then the segments connecting

[1] From the standpoint of *projective geometry*, a parabola is just an ellipse with one focus placed on the line at infinity.

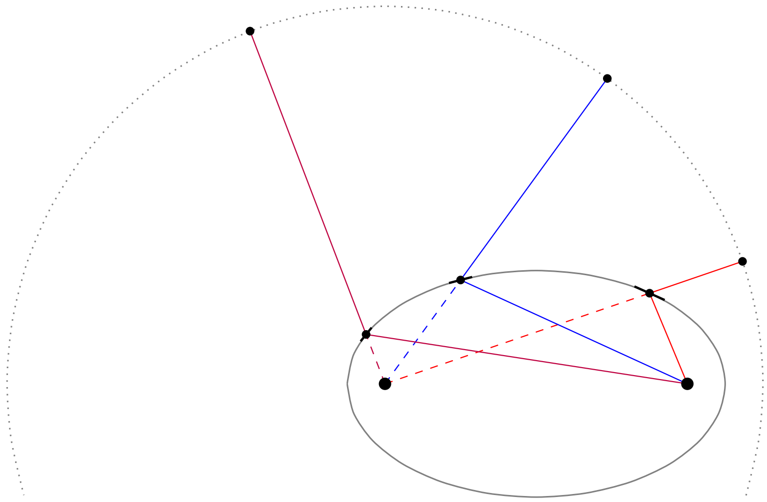

Figure 4: For any point on the ellipse (drawn in black), its distance to the focus on the right and its distance to the circle of radius d are equal. The three colored trajectories within the ellipse meet the ellipse at equal angles.

P to the two foci meet this line at equal and opposite angles of incidence.

Hyperbolas

We briefly mention a few properties of hyperbolas, which can be proven by the same techniques used for ellipses. Just as the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ defines an ellipse, the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ defines a hyperbola with foci on the x-axis at $(\pm\sqrt{a^2+b^2}, 0)$ (while $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ has foci on the y-axis). A hyperbola has the property that every point on the hyperbola has the *difference* of its distances to the two foci equal to $2a$.

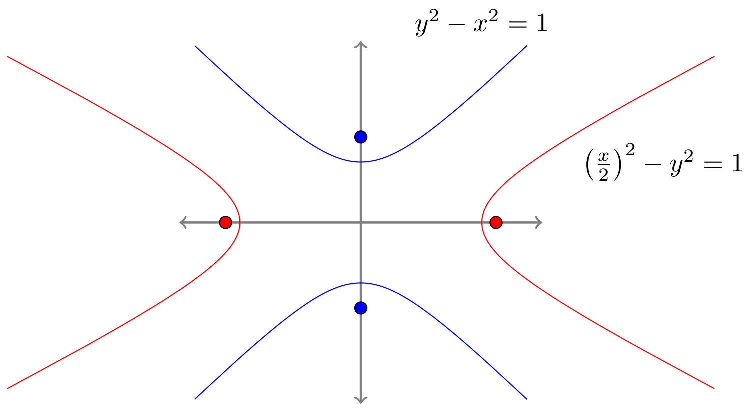

Figure 5: Two hyperbolas shown in red and blue, with their foci accordingly displayed.