Omega Class #1 – Calculating functions

Krishanu Sankar

September 21, 2025

Introduction

Since ancient times, the tools and concepts of mathematics have developed to address practical matters in astronomy, navigation, geography, carpentry, metalworking, naval engineering, cryptography, etc. The ability to calculate certain quantities that depend on other known quantities was and is still important.

Let's consider the following simple examples of calculation. *Calculation* should be considered to mean the following: can you physically construct the number in decimal form to any requested level of precision?

1. Calculate the area of a square whose side length is 8 m.
2. Calculate the side length of a square whose area is 2 m^2.
3. Given a circle of radius 1 m and a chord of this circle which cuts an arc of length 1, calculate the length of the chord.

Which of these would you find easy to calculate by hand? Which are difficult? Why?

Types of functions

A *function* is a mathematical object which encodes a relationship between input values and output values.

$f(x) = x^2 + 5$

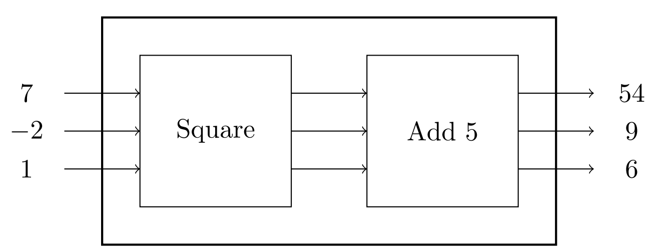

Figure 1: You can think of a function as a machine which transforms inputs to outputs. Above is the function $x^2 + 5$ which consists of two arithmetic steps: squaring and then adding five. The result is shown for a few possible input values.

For example, look at example (1) above. If a square has side length 8, its area is 8^2 ; in general, if the side length is some positive number x, the area is given by the function x^2 ("x-squared"). Functions formed using x, whole numbers, and the four arithmetic operations (addition, subtraction, multiplication, and division) have the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials with integer coefficients. The values of such functions can be easily calculated using arithmetic. Some examples:

$$f(x) = x - 2 \quad ; \quad f(x) = x^2 + 5 \quad ; \quad f(x) = \frac{3}{x+7} \quad ; \quad f(x) = \frac{x^3 + 5x + 8}{2x^4 + 11}$$

What about the second example? Here we must calculate $\sqrt{2}$, which is the square root function $f(x) = \sqrt{x}$ applied to $x = 2$. This is *not* a rational function and it can't be calculated directly by arithmetic. But it is closely related to the first function – they're inverses to each other. If x is a positive real number, then $(\sqrt{x})^2 = x$, and also $\sqrt{x^2} = x$. Since we know how to square a number, we can use this relationship to approximate square roots, as outlined in the next section.

$$f(x) = \sqrt{x - 5}$$

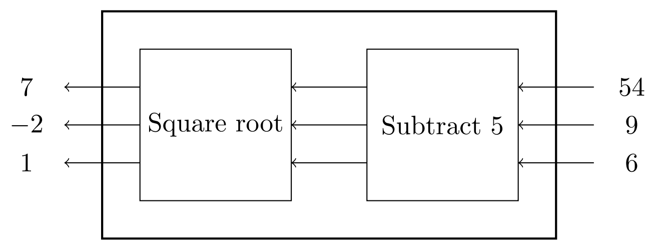

Figure 2: The *inverse* function for a composite function is obtained by undoing the operations in the reverse order. The inverse of adding 5 is subtracting 5 ; the inverse of squaring is taking the square root (assuming the input is a positive number).

Finally, we come to the third example. In the language of trigonometric functions, this value would be $2\sin(1/2)$ (in radians), or $2\sin((90/\pi)^\circ)$. The function $f(x) = \sin(x)$ is a *transcendental* function: it doesn't have any algebraic relationship to functions we know how to compute using arithmetic.

Transcendental functions such as the trigonometric functions $\sin(x)$, $\cos(x)$, and $\tan(x)$ appear in practical applications. For thousands of years, trigonometry was a subfield of astronomy. It was used by sea and desert travelers for navigation, and by priests for predicting celestial phenomena. It was also used by engineers and architects. So people needed a way to calculate these values.

Calculus is a powerful set of techniques for computing transcendental values. More accurately, calculus courses as they are formulated today are an amalgamation and formalization of these techniques developed over the last > 1000 years. Learning the history helps us understand *who* was using these methods, and for *what* purpose.

Two methods to calculate square roots

N.b. For this section, it is recommended that you slow down and try the calculations yourself.

Let's come back to the example of computing $\sqrt{2}$. The decimal expansion of $\sqrt{2}$ will look like $1.d_1d_2d_3d_4\ldots$ for an infinite sequence of digits $d_1, d_2, d_3, d_4, \ldots$. We can solve for these digits one-by-one by maximizing the following inequalities:

$$(1.d_1)^2 = 1^2 + (2 \cdot 1 + 0.d_1)(0.d_1) < 2 \implies d_1 = 4$$

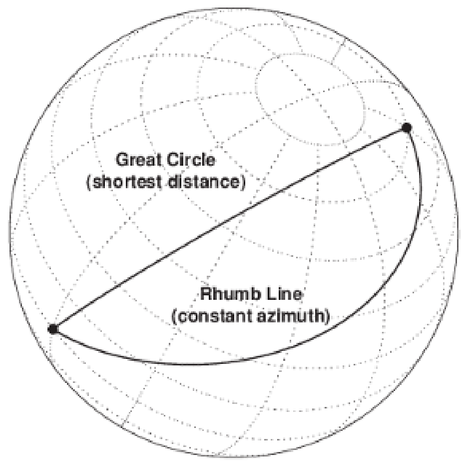

Figure 3: An example problem involving trigonometry: given the latitude and longitude of two points on earth, find the easiest way to navigate between them. The shortest distance "straight" path, it turns out, is difficult to navigate along because the required compass bearing changes continuously.

$$
\begin{gathered}
(1.4d_2)^2 = 1.4^2 + (2 \cdot 1.4 + 0.0d_2)(0.0d_2) < 2 \implies d_2 = 1 \\
(1.41d_3)^2 = 1.41^2 + (2 \cdot 1.41 + 0.00d_3)(0.00d_3) < 2 \implies d_3 = 4 \\
(1.414d_4)^2 = 1.414^2 + (2 \cdot 1.414 + 0.000d_4)(0.000d_4) < 2 \implies d_3 = 2 \\
\vdots
\end{gathered}
$$

This has a formulation similar to long division, as shown below.[1]

$$
\begin{array}{rl}
1\ .\ d_1\ d_2\ d_3 \ldots & \\
1\ .\ 4\ \ 1\ \ 4 \ldots & \\
\overline{)\,2\ .\,00\,00\,00 \ldots} & \\
-\,1 \qquad\qquad & = 1^2 \\
\overline{1\ \ 00} \qquad\quad & \\
-\ \ 80 \qquad\quad & = 10 \cdot 2 \cdot 4 \\
-\ \ 16 \qquad\quad & = 4^2 \\
\overline{4\,00} \qquad & \\
-\ \ 2\,80 \qquad & = 140 \cdot 2 \cdot 1 \\
-\ \ 1 \qquad & = 1^2 \\
\overline{1\,19\,00} \quad & \\
-\ \ 1\,12\,80 \quad & = 1410 \cdot 2 \cdot 4 \\
-\ \ 16 \quad & = 4^2 \\
\overline{6\,04\ 00} & \\
\vdots &
\end{array}
$$

You can use this technique to calculate any square root for arbitrary accuracy. (See `https://www.cantorsparadise.com/the-square-root-algorithm-f97ab5c29d6d` for more explanation.)

[1]Attributed to Aryabhata, 510 CE.

There is another method, which yields a sequence of fractions approximating a square root.[2] Begin with any initial x value, and repeatedly apply the transformation

$$x \mapsto \frac{x + \frac{2}{x}}{2}$$

The idea is that if $x > \sqrt{2}$, then $\frac{2}{x} < \sqrt{2}$, and if $x < \sqrt{2}$, then $\frac{2}{x} > \sqrt{2}$. Therefore, averaging the two numbers should yield something closer to $\sqrt{2}$, and each iterated value should be closer to $\sqrt{2}$ than the last.

If we start with $x = 1$, this process yields subsequent values $\frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \ldots$. This algorithm approximates $\sqrt{2}$ much faster than the previous one ($\frac{665857}{470832}$ and $\sqrt{2}$ agree in the first 11 digits after the decimal point!). Further along in the course you will *prove* various properties of this algorithm, such as the fact that the number of decimal places of accuracy *doubles* with each iteration. We will also use the same method to calculate functions other than square roots.

Both algorithms above for calculating square roots never end. Long division always either ends, or leads to a repeating pattern in the decimal expansion. Meanwhile, the two square root algorithms above only yield an infinite sequence of approximations to the square root – this is because square roots are *irrational* numbers.

[2] This one is attributed to Babylonian texts. It's a special case of Newton's method