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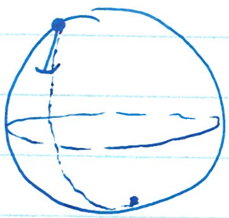
§1. Three Classical Problems

Let n be a positive integer. $X = S^{n-1}$ the unit sphere in $\mathbb{R}^n$.



A tangent vector field is an assignment to every $x \in X$ some $v_x \in \mathbb{R}^n$ s.t. $\langle x, v_x \rangle = 0$

Q: For which n is there an everywhere nonzero tangent vector field?

A: n even $\Rightarrow S^{2m-1}$ is the unit sphere in $\mathbb{C}^m$ and $x \mapsto ix$ works YES

n odd $\Rightarrow$ if there is a tangent vector field, then there's a homotopy $\mathrm{id}_{S^{n-1}} \sim \mathrm{antipodal}_{S^{n-1}}$ by pushing each point x along the great circle in the direction of v_x.

$\deg = 1$ $\deg = -1$

Q: ~~How many~~ Max # of linearly indendent vector fields exist on S^{n-1}?

(This number $+1$) is denoted $\rho(n)$, the n-th Radon-Hurwitz number.

The answer is known, the maximal number is constructed using Clifford algebras, and the proof that this is optimal uses K-theory.

Ex: if n odd, $\rho(n) = 1$. In general, $\rho(n)$ depends on only the max power of 2 dividing n.

Special case: A manifold is parallelizable if its tangent bundle has a basis. (Ex. Lie groups are parallelizable)

Q: Which spheres are parallelizable? (I.e. for what n is $\rho(n) = n$?)

A (Kervaire): S^{n-1} is parallelizable iff $n = 1, 2, 4,$ or 8.

Note: By Gram-Schmidt, we can assume the vector fields are orthonormal.

k orthonormal vector fields $\longleftrightarrow$ $V_{n,k}$ = Stiefel bundle of tangent orthonormal k-frames, $V_{n,k} \downarrow S^{n-1}$ with a section $\exists$

$n-1$ orthonormal vector fields (parallelizable) $\longleftrightarrow$ $O(n-1) \to O(n) \downarrow S^{n-1}$ with a section $\exists$

$$\begin{tikzcd} O(n)\times S^{n-1} \arrow[r] & S^{n-1} \\ S^{n-1}\times S^{n-1} \arrow[u, "?"] \arrow[ur, "\mu"'] & \end{tikzcd}$$

$O(n)$ acts on S^{n-1}.
Thus if S^{n-1} is parallelizable, we get an H-space structure $\mu: S^{n-1}\times S^{n-1} \to S^{n-1}$

Q: Which S^{n-1}'s are H-spaces?

<u>Ans</u>: $n=$	1	2	4	8
units in	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$

Given an H-space structure $S^{n-1}\times S^{n-1} \xrightarrow{\mu} S^{n-1}$ we can perform the <u>Hopf construction</u> to get $S^{n-1} * S^{n-1} \to \Sigma S^{n-1}$

i.e. $S^{2n-1} \xrightarrow{H(\mu)} S^n$

$$\begin{tikzcd} S^{n-1}\times S^{n-1} \arrow[r] \arrow[d] & S^{n-1}\times CS^{n-1} \arrow[d] \\ CS^{n-1}\times S^{n-1} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & S^{n-1} * S^{n-1} \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & CS^{n-1} \arrow[d] \\ CS^{n-1} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \Sigma S^{n-1} \end{tikzcd}$$

In general for any map $\eta: S^{2n-1} \to S^n$, $(n \geq 2)$

form the cell complex $X = S^n \cup_\eta e^{2n}$. Then $H^*(X;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & *=0,n,2n \\ 0, & o/w \end{cases}$

$$\begin{tikzcd} S^{2n-1} \arrow[r, "\eta"] \arrow[d, "\text{bdry}"'] & S^n \arrow[d] \\ e^{2n} \arrow[r] \arrow[ur, phantom, "\lrcorner"] & X \end{tikzcd}$$

Let $x \in H^n$ and $y \in H^{2n}$ be generators
Then $x^2 = \pm \alpha y$ for some $\alpha \in \mathbb{Z}$. $\alpha =$ "Hopf invariant"

<u>Fact</u>: If $\eta = H(\mu)$ for an H-space structure μ, then $\alpha = \pm 1$.

<u>Q</u>: Is there an element of Hopf invariant one in $\pi_{2n-1}(S^n)$?

<u>A (Adams)</u>: There is if and only if $n = 1,2,4,8$

S^{n-1} parallelizable $\Rightarrow$ $\exists$ H-space structure on S^{n-1} $\Rightarrow$ $\exists\, \eta \in \pi_{2n-1}(S^n)$ with Hopf invariant $= \pm 1$

Prop: Let $n \geq 2$ and suppose $\exists \eta \in \pi_{2n-1}(S^n)$ with Hopf invariant $\equiv 1 \pmod 2$. Then n is a power of 2.

Facts we'll need: The mod 2 Steenrod algebra $\underbrace{\mathbb{F}_2\langle Sq^1, Sq^2, Sq^3, \ldots\rangle}^{\text{non-commuting generators}} / \text{Adem relations}$

is generated as an algebra by $Sq^1, Sq^2, Sq^4, Sq^8, \ldots$

For any space X and class $x \in H^n(X; \mathbb{F}_2)$, the cup square x^2 is equal to $Sq^n(x)$.

Pf: Suppose $\eta \in \pi_{2n-1}(S^n)$ has mod 2 Hopf invariant 1. Let $X = e^{2n} \cup_\eta S^n$. Let $x \in H^n(X; \mathbb{F}_2)$ be the generator. Then by assumption, $x^2 = Sq^n(x)$ is nonzero. However, because $H^*(X; \mathbb{F}_2) \cong 0$ for $* = n+1, \ldots, 2n-1$, it follows that $Sq^i(x) = 0$ for $i = 1, 2, \ldots, n-1$. Therefore, Sq^n is indecomposable in the mod 2 Steenrod algebra. So n is a power of 2. ∎

To prove $n = 1, 2, 4, 8$ are the only ones ~~requires~~ we'll use K-theory.

§2: Power Operations

Let A be an associative, commutative monoid. This means the product

$$\begin{tikzcd} A \arrow[r, "\Delta"] & A^{\times n} \arrow[r, "\mu"] \arrow[d] & A \\ & A^{\times n}/\Sigma_n \arrow[ur, dashed] & \end{tikzcd}$$

factors as shown.

n-th power map.

Σ_n = symmetric gp on n letters

In topology, we don't demand that associativity and commutativity are strict.

A space A is an E_∞-algebra if it has an action of the E_∞-operad.

$$B\Sigma_n \times A \xrightarrow{\Delta} E\Sigma_n \times_{\Sigma_n} A^{\times n} \xrightarrow{\mu} A$$

There is an entire space of multiplications.

The space $E\Sigma_n = \mathrm{Conf}_n(\mathbb{R}^\infty)$ configs of n pts in $\mathbb{R}^\infty$ of multiplications is contractible

"Associative and commutative up to coherent homotopy."

Example: For any pointed X, $\Omega^k \Sigma^k X$ is an E_k-algebra and $\Omega^\infty \Sigma^\infty X := \operatorname{colim}_{k \to \infty} \Omega^k \Sigma^k X$ is an E_∞-algebra.

Ex: π_0 of an E_1-algebra is an assoc. monoid

π_0 of an E_2-algebra is an assoc., comm. monoid.

$$\underbrace{B\Sigma_n \times A \xrightarrow{\Delta} E\Sigma_n \times_{\Sigma_n} A^{\times n} \xrightarrow{\mu} A}_{n\text{-th power map.}}$$

The generalized cohomology $E^*(B\Sigma_n)$ gives rise to power operations in E-cohomology.

Ex: When $E^* = H^*(-;\mathbb{Q})$, there is no nontrivial cohomology, and no power operations. This corresponds to the fact that the cup product on $H^*(X;\mathbb{Q})$ can be made strictly graded commutative on $C^*(X;\mathbb{Q})$.

Ex: When $E^* = H^*(-;\mathbb{F}_p)$, lots of power operations. These are all generated from the $n=p$ case. "Deformations / derived functors of the p-fold cup power."

Vague: $H^*(B\Sigma_p;\mathbb{F}_p)$ is concentrated in degrees $0, 2(p-1)-1, 2(p-1), 4(p-1)-1, 4(p-1), 6(p-1)-1, \ldots$

Acting on a class of degree $2n$, the class of deg

$2n(p-1)$	is $(-)^p$ (i.e. trivial deformation)
$2n(p-1)-1$	β
$2(n-1)(p-1)$	P^1
$2(n-1)(p-1)-1$	$P^1\beta$
$2(n-2)(p-1)$	P^2
$2(n-2)(p-1)-1$	$P^2\beta$
$\vdots$	$\vdots$

The Adem relations come from considering
$$\Sigma_p \times \cdots \times \Sigma_p \hookrightarrow \Sigma_{p^2}$$

§3: K-theory.

$U(n) = \{\text{unitary } n\times n \text{ matrices}\}$

$BU(n)$: Grassmannian of n-planes in $\mathbb{C}^\infty$. $BU(i)\times BU(j) \to BU(i+j)$ by interleaving coordinates $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \cong \mathbb{C}^\infty$.

For nice spaces X, $[X, BU(n)] \cong \{\text{complex vector bundles over } X \text{ of rank } n\}/\text{iso}$

$X \to X\times X \xrightarrow{(V,W)} BU(i)\times BU(j) \to BU(i+j)$ gives Whitney sum $V\oplus W$.

The inclusion of the first n coords $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ gives $\mathbb{C}\oplus\mathbb{C}^\infty \cong \mathbb{C}^\infty$ shift map

$$\operatorname{hocolim}(\cdots \to BU(n-1) \to BU(n) \to BU(n+1) \to \cdots) =: BU$$

Then BU is an E_∞-algebra. $K^0(X) := K(X) = [X, BU] = $ Grothendieck group of $\{\mathbb{C}\text{-v.b.'s over } X\}/\text{iso}$.

Define $K^{-n}(X) := K^0(\Sigma^n X)$. Then the functors $\{K^{-n}(-)\}_{n\in\mathbb{Z}}$ form a cohomology theory.

$KU = \{\mathbb{Z}\times BU, U, \mathbb{Z}\times BU, U, \ldots\}$ is a spectrum which represents this coh. thy.

For all $*\in\mathbb{Z}$, $\pi_* KU \cong \begin{cases} \mathbb{Z} & * \text{ even} \\ 0 & * \text{ odd.} \end{cases}$ (By Bott periodicity, $\Omega U \simeq \mathbb{Z}\times BU$.)

It's automatically true that $\Omega BU \simeq U$.

$BU(i) \wedge BU(j) \longrightarrow BU(ij)$ coming from $\mathbb{C}^\infty \otimes_{\mathbb{C}} \mathbb{C}^\infty \cong \mathbb{C}^\infty$

classifies external tensor product
$$\begin{matrix} V \\ \downarrow \\ X \end{matrix} \otimes \begin{matrix} W \\ \downarrow \\ Y \end{matrix} \rightsquigarrow \begin{matrix} V \otimes W \\ \downarrow \\ X \times Y \end{matrix}$$

This product makes KU into a ring spectrum, and $K(-)$ into a ring
$$K(X) \otimes K(Y) \to K(X \times Y)$$

Atiyah-Segal Completion Theorem: $K^*(BG) \cong \mathrm{Rep}_{\mathbb{C}}(G)^\wedge_I$, I = augmentation ideal containing formal differences $[V]-[W]$ with $\dim V = \dim W$

For a complex vector space V, the n-fold tensor power $V^{\otimes n}$ can be decomposed into a direct sum of components $\{V^{\otimes n}(\pi)\}$ for each complex irrep. π of the symmetric group Σ_n.

Ex: Trivial rep $\rightsquigarrow \mathrm{Sym}^n V$

Sign rep $\rightsquigarrow \Lambda^n V$.

Power operations on K-theory should come from $K^*(B\Sigma_n)$, and somehow the Adams operations ψ^n, which comes from the n-th exterior power construction, generate everything.

Q: Understand the construction & why they generate. $BU \circlearrowleft \psi^n$

Q: One can write formulas for their action on K-theory. Explain these.

Q: Learn the classical applications.

Q: Adams operations in algebraic K-theory (see work of Quillen)? Commutative and nilpotent K-theory?

Q: Power operations on other cohomology theories. Ex: Morava E-theories, Complex cobordism.

§1: Characters of $GL_n(\mathbb{C})$

Let $GL_n(\mathbb{C}) = \{n\times n$ invertible matrices over the complex numbers$\}$.

Let $E \cong \mathbb{C}^n$ denote the standard n-dimensional complex rep'n.

Imagine that we consider a generic element $T \in GL_n(\mathbb{C})$ with eigenvalues $t_1, t_2, \dots, t_n$. Then the trace of T is $t_1+t_2+\dots+t_n$.

We say E has character polynomial $\chi_E(t_1,\dots,t_n) = t_1+t_2+\dots+t_n$

A <u>non-stupid example</u>: Pick an integer $k\geq 2$. Then $\underbrace{E\otimes_{\mathbb{C}} E\otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} E}_{k \text{ times}} = E^{\otimes k}$

is a representation of $GL_n(\mathbb{C})$. The trace of the generic element T is given by the polynomial

$$\chi_{E^{\otimes k}}(t_1,t_2,\dots,t_n) = (t_1+t_2+\dots+t_n)^k$$

Notice $\chi_{E^{\otimes k}}$ has <u>degree k</u> and is <u>symmetric</u>, i.e Σ_n-invariant

<u>More examples</u>: • $\Lambda^2 E = \langle e_i\otimes e_j : 1\leq i<j\leq n\rangle = \langle e_i\otimes e_j : i,j\rangle / (e_i\otimes e_j = -e_j\otimes e_i)$

then $\chi_{\Lambda^2 E} = \sum_{i<j} t_it_j$

• $Sym^2 E = \langle e_i\otimes e_j : i,j\rangle / (e_i\otimes e_j = e_j\otimes e_i) = \langle e_i\otimes e_j : 1\leq i\leq j\leq n\rangle$

then $\chi_{Sym^2 E} = \sum_{i\leq j} t_it_j$

• $E^{\otimes 2} \cong \Lambda^2 E \oplus Sym^2 E$ and $\left(\sum_i t_i\right)^2 = \sum_{i<j} t_it_j + \sum_{i\leq j} t_it_j$.

What about $E^{\otimes k}$? It has an action of the symmetric group Σ_k.

<u>Example</u>: $E^{\otimes 2} \cong \underbrace{Sym^2 E}_{\Sigma_2 \text{ acts trivially}} \oplus \underbrace{\Lambda^2 E}_{\Sigma_2 \text{ acts as the sign representation.}}$

In general, $E^{\otimes k}$ decomposes into a direct sum of subrep'ns (each with $GL_n\mathbb{C} \times \Sigma_k$ action)

<u>Fact</u>: $\left\{ \begin{array}{c} \text{Irreps of } \Sigma_k \\ \text{over } \mathbb{C} \end{array} \right\} \xleftrightarrow{\sim} \left\{ \begin{array}{c} \text{partitions } \pi \\ \text{of } k, \end{array} \right\}$ In fact, the irreps are all defined over $\mathbb{Q}$

We don't actually need this fact. Let $\{V_\pi\}_\pi$ denote the set of irreducible rep'ns over Σ_k (up to isomorphism)

Let $\pi(E) := \mathrm{Hom}_{\Sigma_k}(V_\pi, E^{\otimes k})$.

Then $E^{\otimes k} \cong \bigoplus_\pi V_\pi \otimes \pi(E)$. (Note: $\pi(E)$ is just a complex vector space whose dimension = # of copies of V_π in the direct sum decomposition of $E^{\otimes k}$. The number k is implicit, because π is a partition of k.)

Each summand $V_\pi \otimes \pi(E)$ is $\underset{\Sigma_k}{\text{Rep of}} \otimes \underset{GL_n(\mathbb{C})}{\text{Rep of}}$.

For each π, the $GL_n(\mathbb{C})$-rep'n $\pi(E)$ has an associated character polynomial $\chi_{\pi(E)}$. We have the formula

$$(t_1 + \cdots + t_n)^k = \chi_{E^{\otimes k}} = \sum_\pi \dim(V_\pi) \cdot \chi_{\pi(E)}.$$

<u>Example</u>: $k=3$. Then Σ_3 has 3 irreps: $\underset{\text{1-dim'l}}{\mathbb{1}}$, $\underset{\text{1-dim'l}}{\mathrm{sgn}}$, $\underset{\text{2-dim'l}}{V}$

$$(t_1 + \cdots t_n)^3 = \sum_{i \le j \le k} t_i t_j t_k + \sum_{i<j<k} t_i t_j t_k + 2\left(\sum_{i<j} (t_i^2 t_j + t_i t_j^2) + \sum_{i<j<k} t_i t_j t_k \right).$$

Fun <u>fact</u>: As k varies, this construction builds all of the finite dim. irreps of $GL_n\mathbb{C}$.

§2: Representation rings

A more refined statement is as follows. Consider

$$\mathrm{Sym}_k[t_1,\dots,t_n] = \left\{ \begin{array}{l} \text{symmetric} \\ \text{polynomials in variables } t_1,\dots,t_n \\ \text{over the complex numbers} \end{array} \right\}$$

$R(\Sigma_k)$ = Complex rep'n ring of Σ_k. This is a free abelian group on the irreps $[V_\pi]$ under direct sum. The product comes from tensor product over $\mathbb{C}$.

{ can take "virtual representations" like $-[V_\pi]$.

Then there is an element $\Delta_{n,k} \in \mathrm{Sym}_k[t_1,\dots,t_n] \otimes R(\Sigma_k)$ defined by

$$\Delta_{n,k} = \sum_{\pi} (\chi_{\pi(E)} \otimes V_\pi)$$

and if we apply the map

$$\begin{array}{ccc} \mathrm{Sym}_k[t_1,\dots,t_n] \otimes R(\Sigma_k) & & \Delta_{n,k} \\ \downarrow \text{dimension} & & \downarrow \\ \mathrm{Sym}_k[t_1,\dots,t_n] \otimes \mathbb{Z} \cong \mathrm{Sym}[t_1,\dots,t_n] & & (t_1+\dots+t_n)^k. \end{array}$$

Even more structure

Let k,ℓ be positive integers. There's an obvious inclusion $\Sigma_k \times \Sigma_\ell \hookrightarrow \Sigma_{k+\ell}$

Pulling back along this inclusion, $R(\Sigma_{k+\ell}) \longrightarrow R(\Sigma_k \times \Sigma_\ell) \cong R(\Sigma_k) \otimes R(\Sigma_\ell)$.

Thus, the direct sum $\bigoplus_{k\geq 0} R(\Sigma_k)$ is a coalgebra, b/c $R(\Sigma_n) \longrightarrow \bigoplus_{i+j=n} (R(\Sigma_i) \otimes R(\Sigma_j))$.

Dualizing: let $R_*(\Sigma_k) := \mathrm{Hom}(R(\Sigma_k), \mathbb{Z})$

let $R_* := \bigoplus_{k\geq 0} R_*(\Sigma_k)$. Then R_* is an algebra.

Note that $\Delta_{n,k} \in \mathrm{Sym}_k[t_1,\dots,t_n] \otimes R(\Sigma_k)$ corresponds to ~~an~~ a homomorphism

$\Delta_{n,k}^*: R_*(\Sigma_k) \longrightarrow \mathrm{Sym}_k[t_1,\dots,t_n]$ of abelian groups.

$$\varphi \longmapsto \sum_{\pi} (\chi_{\pi(E)} \cdot \underbrace{\varphi(V_\pi)}_{\text{an integer.}})$$

Proposition: Define $\Delta_n^* : R_* \longrightarrow \mathrm{Sym}[t_1,\dots,t_n]$. Then Δ_n^* is a <u>ring</u> homomorphism.

$$R_* \cong \bigoplus_{k\geq 0} R_+(\Sigma_k) \qquad \mathrm{Sym}[t_1,\dots,t_n] \cong \bigoplus_{k\geq 0} \mathrm{Sym}_k[t_1,\dots,t_n]$$

Note: R_* has no dependence on n.

Proposition:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & \mathrm{Sym}[t_1,t_2,t_3] \arrow[d] & t_3 \arrow[d, mapsto] \\
 & \mathrm{Sym}[t_1,t_2] \arrow[d] & 0 \quad t_2 \arrow[d, mapsto] \\
R_* \arrow[r, "\Delta_1"] \arrow[ur, "\Delta_2"] \arrow[uur, "\Delta_3"] \arrow[uuur] & \mathrm{Sym}[t_1] & 0
\end{tikzcd}$$

and $R_* \cong \varprojlim_n \mathrm{Sym}[t_1,\dots,t_n]$ $\boxed{\text{Call this iso } \Phi.}$

$\cong \mathrm{Sym}[t_1,t_2,\dots]$

symmetric "polynomials" in countably infinitely many variables.

The first proposition essentially is as follows. If we view $R(\Sigma_k)$ as a ring of characters of Σ_k (by applying the trace) then the element $\Delta_{n,k} \in \mathrm{Sym}_k[t_1,\dots,t_n] \otimes R(\Sigma_k)$

is the function $(t_1,\dots,t_n), g \longmapsto \mathrm{Trace}(gT^{\otimes k})$

Now the fact that Δ_n^* is multiplicative holds because if $g \in \Sigma_k$ and $h\in\Sigma_\ell$, then

$$\mathrm{Trace}\left((g,h)\cdot T^{\otimes(k+\ell)}\right) = \mathrm{Trace}(g\cdot T^{\otimes k})\cdot \mathrm{Trace}(g\cdot T^{\otimes \ell}).$$

Now define specific elements. Let $\mathbb{1}$ = 1-dim'l trivial rep of Σ_k (over $\mathbb{C}$)

sgn = 1-dim'l sign rep of Σ_k (over $\mathbb{C}$)

Let $\sigma^k : R(\Sigma_k) \longrightarrow \mathbb{Z}$ be the indicator function for $\mathbb{1}$

λ^k: ——————————— sgn

Then $\Phi(\sigma^k) = \sum_{i_1\leq\cdots\leq i_k} (t_{i_1}\cdots t_{i_k})$. Call this polynomial h_k

$\Phi(\lambda^k) = \sum_{i_1<\cdots<i_k} (t_{i_1}\cdots t_{i_k}) = e_k$ "k-th elementary symmetric polynomial."

Note that $\mathrm{Sym}[t_1, t_2, \ldots]$ is a polynomial ring on $e_0, e_1, e_2, e_3, \ldots$

It's also a polynomial ring on $h_0, h_1, h_2, \ldots$

Thus, $R_* \cong \mathrm{Poly}(\sigma^0, \sigma^1, \sigma^2, \ldots) \cong \mathrm{Poly}(\lambda^0, \lambda^1, \lambda^2, \ldots)$

3: The Adams operations:

$\psi^k \in R_*$ is the element mapping to the polynomial $(t_1^k + t_2^k + \cdots)$.

Let m_k denote the polynomial $t_1^k + t_2^k + \cdots$

Newton polynomials: $Q_k(e_1, e_2, \ldots, e_k) = m_k$.

Then $Q_k(\lambda^1, \lambda^2, \ldots, \lambda^k) = \psi^k$.

Example: $k=2$. Then $e_1^2 - 2e_2 = m_2$

$(\lambda^1)^2 - 2\lambda^2 = \psi^2$

So, $\psi^2(E) = [E^{\otimes 2}] - 2[\Lambda^2 E]$

$= [\mathrm{Sym}^2 E] - [\Lambda^2 E]$ is an element of the Grothendieck group of representations of $GL_n(\mathbb{C})$.

virtual dimension $= \frac{n(n+1)}{2} - \frac{n(n-1)}{2}$
$= n$.

Prop: In general, $\psi^k : R(GL_n(\mathbb{C})) \longrightarrow R(GL_n(\mathbb{C}))$ is a ring homomorphism.

Note: $GL_n(\mathbb{C})$ can be replaced by any group G. If we're talking about n-dimensional rep'ns of G, then $G = GL_n(\mathbb{C})$ is the universal case.

Let's think of $\psi^k : R(G) \longrightarrow R(G)$

Proof: The fact that $\psi^k[E \otimes F] = \psi^k[E]\,\psi^k[F]$ is fairly straightforward.

Let $\dim E = m$, and a typical element of $GL_m\mathbb{C}$ has eigenvalues $t_1, \ldots, t_m$

$\dim F = n$ —— $GL_n\mathbb{C}$ —— $s_1, \ldots, s_n$

Then $\psi^k[E]$ has character polynomial $(t_1^k + \cdots + t_m^k)$

$\psi^k[F]$ —— $(s_1^k + \cdots + s_n^k)$

$E \otimes F$ has an action of $GL_m\mathbb{C} \times GL_n\mathbb{C}$ w/ ~~character polynom~~ eigenvalues $\{t_i s_j\}_{\substack{1 \le i \le m \\ 1 \le j \le n}}$

So $\psi^k[E \otimes F]$ has character polynomial $\sum_{i,j} t_i^k s_j^k = (\sum_i t_i^k)(\sum_j s_j^k)$

Thus $\psi^k[E \otimes F]$ and $\psi^k[E]\,\psi^k[F]$ have the same character polynomials.

The proof that $\psi^k([E]+[F]) = \psi^k[E] + \psi^k[F]$ requires more work, or so Atiyah has us believe. I don't see why the following argument doesn't work.

Let E be a complex rep'n of G with dimension m. I.e., $\varphi_E : G \longrightarrow GL(E) \cong GL_m(\mathbb{C})$.
~ F ———————————— n $\varphi_F : G \longrightarrow GL(F) \cong GL_n(\mathbb{C})$.

Pick any element $g \in G$. Suppose that $\varphi_E(g)$ has eigenvalues $t_1, \dots, t_m$
and $\varphi_F(g)$ ——————— $s_1, \dots, s_n$.

By definition, $\mathrm{Trace}_{\psi^k[E]}(g) = t_1^k + \dots + t_m^k$ and $\mathrm{Trace}_{\psi^k[F]}(g) = s_1^k + \dots + s_n^k$

So $\mathrm{Trace}_{\psi^k[E] + \psi^k[F]}(g) = (t_1^k + \dots + t_m^k) + (s_1^k + \dots + s_n^k)$

On the other hand, g acts on $E \oplus F$ with eigenvalues $t_1, \dots, t_m, s_1, \dots, s_n$
(i.e. look at $G \xrightarrow{(\varphi_E, \varphi_F)} GL(E) \times GL(F) \xrightarrow{\begin{pmatrix}\blacksquare & 0 \\ 0 & \blacksquare\end{pmatrix}} GL(E \oplus F)$)

So $\mathrm{Trace}_{\psi^k[E \oplus F]}(g) = (t_1^k + \dots + t_m^k) + (s_1^k + \dots + s_n^k)$

Therefore, $\psi^k[E] + \psi^k[F]$ and $\psi^k[E \oplus F]$ have the same character.
as rep'ns of G. It is standard that they thus are equal in $R(G)$. ∎

Interlude:

<u>A brief Background on representation theory</u>: Let G be a finite group, and let $\mathbb{F}$ be a field.
For this discussion we use $\mathbb{F} = \mathbb{C}$.

A finite dimensional representation of G is a finite-dimensional vector space over $\mathbb{C}$ with an action of G.
A subrepresentation of V is a subspace $W \subseteq V$ such that $\forall w \in W$, $\forall g \in G$, $gw \in W$
A representation V is <u>irreducible</u> if it has no subrepresentations besides 0 and V.

<u>Prop</u>: Any f.d. representation V decomposes into subrepresentations $V \cong V_1 \oplus \dots \oplus V_n$
with each V_i irreducible.

<u>Proof</u>: Suppose $V_1 \subset V$ is irreducible. Define $\varphi : V \longrightarrow V_1$, $v \longmapsto \frac{1}{|G|}\sum_{g \in G} g\pi(g^{-1}v)$. Then φ acts by the identity on V_1. ∎
Pick a vector space projection $\pi : V \longrightarrow V_1$. You can easily check φ is G-equivariant
So $V \cong V_1 \oplus \ker \varphi$.

Now just use induction on $\dim(V)$.

Representations of G can be understood by their <u>character</u>. I.e. if E is a rep'n of G, consider its character χ_E, which is the function

$$\chi_E : G \longrightarrow \mathbb{C}$$
$$g \longmapsto \mathrm{Trace}(\varphi_E(g)).$$

Since $\mathrm{Trace}(ABA^{-1}) = \mathrm{Trace}(B)$, it follows that χ_E is constant over conjugacy classes of G. That is, the character is a function that is additive and multiplicative

$$\chi_{(-)} : \mathrm{Rep}(G) \longrightarrow Cl(G;\mathbb{C})$$

complex vector space of functions $\{\text{Conj classes of } G\} \longrightarrow \mathbb{C}$.

<u>Fact</u>: ~~The~~ Let the irreducible representations of G be $V_1, V_2, \dots, V_n$.

Then $\chi_{V_1}, \dots, \chi_{V_n}$ form an orthonormal basis of $Cl(G;\mathbb{C})$, where the inner product on $Cl(G;\mathbb{C})$ is given by

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_1(g)}\, \chi_2(g).$$

Thus, the representation theory (over $\mathbb{C}$) of a finite group G can be understood by studying $Cl(G;\mathbb{C})$.

I think something analogous can be said when G is a complex Lie group.

For $G = GL_n\mathbb{C}$, finite dimensional virtual representations correspond to those functions $Cl(GL_n\mathbb{C}, \mathbb{C})$ given by symmetric polynomials on $t_1, \dots, t_n$.

(Remember two diagonalizable matrices are conjugate $\Leftrightarrow$ same eigenvalues with multiplicity)

As defined, if E is a rep'n of G with dimension n, and $g \in G$ acts on E with eigenvalues $t_1, \dots, t_n$, then $\psi^k[E]$ is a virtual rep'n of dimension n and $g \curvearrowright \psi^k[E]$ with eigenvalues $t_1^k, \dots, t_n^k$. So we deduce

- $\psi^k[E \otimes F] = \psi^k[E]\,\psi^k[F]$
- $\psi^k[E \oplus F] = \psi^k[E] + \psi^k[F]$
- $\psi^k(\psi^\ell([E])) = \psi^{k\ell}[E]$
- If L is a rep'n of dimension 1, $\psi^k(L) = L^{\otimes k}$
- $\psi^1 = id$.

- Notice that if p is prime, $(t_1 + \dots + t_n)^p = (t_1^p + \dots + t_n^p) + p \cdot (\text{stuff})$

Therefore, $(\psi^1)^p = \psi^p + p\,\theta^p$ for an operator θ^p.

In particular, $(\psi^1)^p \equiv \psi^p \pmod p$.

Proof of Hopf Invariant One

As a reminder, Definition: $(n\geq 2)$

~~Theorem (Adams, Atiyah)~~ Let $f: S^{2n-1} \longrightarrow S^n$ be any map. Form the CW complex $X = S^n \cup_f e^{4n}$.

$$H^*(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & *=0,n,2n \\ 0 & \end{cases}$$

Let $x \in H^n(X;\mathbb{Z})$ and $y \in H^{2n}(X;\mathbb{Z})$ be the generators. Then $x^2 = ay$ for some $a \in \mathbb{Z}$ (well-defined up to sign). The number a is called the <u>Hopf invariant</u> of the class $f \in \pi_{2n-1}(S^n)$.

<u>Theorem (Adams, Atiyah)</u>: If f is an element of $\pi_{2n-1}(S^n)$ with odd Hopf invariant, then $n = 2, 4,$ or 8.

<u>Note</u>: The above theorem implies that S^{n-1} has an H-space structure for only $n = 1,2,4,8$. These are thus the only parallelizable spheres.

<u>Note</u>: The Hopf invariant is an invariant which detects a nontrivial class in the homotopy groups of spheres. It also tell us something about geometry. It turns out that many other geometric structures are tied to specific elements in the homotopy groups of the spheres, and this is a major reason to calculate these groups.

§1. Formal properties of topological K-theory

For a space X, <u>pointed</u>, paracompact Hausdorff etc...,

$K(X)$ = Grothendieck group of f.d. vector bundles on X. Product given by $\otimes$.
$\tilde{K}(X) \subset$ = the <u>ideal</u> contain elements of virtual dimension zero at the basepoint.

You can think of $\tilde{K}(X) = \ker\left(K(X) \xrightarrow{\dim} \underset{\mathbb{Z}}{K(\text{basept})}\right)$

This map splits: $\mathbb{Z} \longrightarrow K(X)$ given by the trivial bundle. So $K(X) \cong \tilde{K}(X) \oplus \mathbb{Z}$.

Since $\tilde{K}(X)$ is an <u>ideal</u>, $K(X) \supset \tilde{K}(X) \supset \tilde{K}(X)^2 \supset \cdots$ filtration.

图

$$\boxed{\begin{tikzcd} & \tilde{K}(X)\otimes\tilde{K}(Y) \arrow[dl, dashed, "\wedge"'] \arrow[d, "\otimes"] \arrow[dr, "0"] & \\ \tilde{K}(X\wedge Y) \arrow[r] & \tilde{K}(X\times Y) \arrow[r, "f^*"] & \tilde{K}(X\vee Y) \end{tikzcd}}$$

If $\begin{smallmatrix} V \\ \downarrow \\ X \end{smallmatrix}$ and $\begin{smallmatrix} W \\ \downarrow \\ Y \end{smallmatrix}$ are vector bundles, then you can construct $\begin{smallmatrix} V\otimes W \\ \downarrow \\ X\times Y \end{smallmatrix}$ and pullback along the inclusion $f: X\vee Y \hookrightarrow X\times Y$ to get $f^*(V\otimes W)$.

If $\dim(V) = \dim(W) = 0$, then $f^*(V\otimes W) = 0$, because

- Restriction of $V\otimes W$ to $X\times\{pt\}$ is $V\otimes 0 = 0$
- Restriction of $V\otimes W$ to $\{pt\}\times Y$ is $0\otimes W = 0$

Thus we obtain the commutative diagram in the box above.

Since $\tilde{K}(-)$ takes cofiber sequences to exact sequences, we obtain the lift

$$\wedge : \tilde{K}(X)\otimes\tilde{K}(Y) \longrightarrow \tilde{K}(X\wedge Y).$$

§2. Bott periodicity

Notice that $S^2 \cong \mathbb{C}P^1 := \mathbb{C}^2 - \{(0,0)\} \big/ \mathbb{C}^\times$. Let $\begin{smallmatrix} L \\ \downarrow \\ \mathbb{C}P^1 \end{smallmatrix}$ denote the canonical complex line bundle.

Let $1 = [\mathbb{1}] \in K(\mathbb{C}P^1)$ denote the class of the trivial complex line bundle.

<u>Fact:</u> Let $\Delta: \mathbb{C}P^\infty \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$ be the diagonal map. Then

$$\mathbb{1} \oplus \Delta^*(L\otimes L) = L\oplus L. \quad (\text{In } K(\mathbb{C}P^1), \text{ we have } [L]^2 + 1 = 2[L], \text{ or } ([L]-1)^2 = 0)$$

<u>Fact:</u> $[L]$ generates $K(\mathbb{C}P^1)$ as a ring.

So, let $x_1 = L - 1$ denote the element of $\tilde{K}(S^2) \cong \tilde{K}(\mathbb{C}P^1)$. We have $x_1^2 = 0$.

That is, $\boxed{K(S^2) \cong \mathbb{Z}[x_1]/(x_1^2)}$ and $\boxed{\tilde{K}(S^2) \cong \mathbb{Z}x_1}$ (Note: This is also due to the fact that $\pi_2 BU \cong \mathbb{Z}$ with generator x_1)

Observe that $\boxed{\Psi^k(x_1) = \Psi^k(L-1) = L^k - 1 = (1+x_1)^k - 1 = kx_1,}$ $\boxed{\text{since } x_1^2 = 0.}$

<u>Bott periodicity</u> says that

- For every n, $K(S^{2n-1}) \cong \mathbb{Z}$ and $\tilde{K}(S^{2n-1}) \cong 0$
- For every n, $\boxed{K(S^{2n}) \cong \mathbb{Z}[x_n]/(x_n^2)}$ and $\boxed{\tilde{K}(S^{2n}) \cong \mathbb{Z}x_n}$ for classes x_n.
- The classes x_n are related via

$$\begin{array}{ccc} \tilde{K}(S^2)\otimes\cdots\otimes\tilde{K}(S^2) & \xrightarrow{\wedge} & \tilde{K}(S^{2n}) \\ x_1 \otimes\cdots\otimes x_1 & \longmapsto & x_n. \end{array}$$

Thus, $\boxed{\Psi^k(x_n) = k^n x_n}$

Note: An equivalent formulation of the statements above is to say the homotopy of BU as a ring is given by

$$\pi_*(BU) \cong \mathbb{Z}[x_1, x_2, x_3, \dots]/(x_1^n = x_n) = \mathbb{Z}[x_1], \quad \deg(x_1) = 2.$$

Over $\mathbb{Q}$, one can get the group structure using the fiber sequences

$$\begin{array}{ccccccccccc} U(1) & \to & U(2) & \to & U(3) & \to & \cdots & \to & U(n-1) & \to & U(n) \to \cdots \\ \| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \underset{\text{in } \pi_1}{\underbrace{S^1}} & & \underset{\text{in } \pi_3}{\underbrace{S^3}} & & \underset{\text{in } \pi_5}{\underbrace{S^5}} & & & & \underbrace{S^{2n-3}} & & \underbrace{S^{2n-1}} \end{array}$$

to get classes in $\pi_1(U)$, $\pi_3(U)$, $\pi_5(U)$, ... which give $x_1, x_2, x_3, \dots$

But to show there is no torsion requires harder calculations. There are numerous beautiful proofs of Bott periodicity; we simply use it.

§3. The Chern character

Complex line bundles are classified by maps to $\mathbb{CP}^\infty$. But observe $\mathbb{CP}^\infty \simeq BU(1) \simeq \underset{K(\mathbb{Z},2).}{\overset{\|}{B(B\mathbb{Z})}}$

Thus,

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{Iso classes of} \\ \text{line bundles} \\ \text{on } X\end{array}\right\} & \xrightarrow{c_1(-)} & H^2(X;\mathbb{Z}) \\ \| & & \| \\ [X, \mathbb{CP}^\infty] & \xrightarrow{\cong} & [X, K(\mathbb{Z},2)] \end{array}$$

By definition, ~~It is easily checked that~~ tensor product of line bundles comes from the product $BU(1) \times BU(1) \xrightarrow{\otimes} BU(1)$.

$$\begin{array}{ccccccc} X & \xrightarrow{\Delta} & X \times X & \xrightarrow{(L_1, L_2)} & BU(1) \times BU(1) & \xrightarrow{\otimes} & BU(1) \\ & & & & \| & & \| \\ & & & & K(\mathbb{Z},2) \times K(\mathbb{Z},2) & \xrightarrow{+} & K(\mathbb{Z},2) \end{array}$$

Def: $c_1(-)$ is the Chern class.

thus, $\boxed{c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)}$

We'd like $\otimes$ to be taken to products, not sums. So

Def: Define $ch(-): K(X) \longrightarrow H^*(X;\mathbb{Q})$, the Chern character, as follows.

- For a line bundle L, $ch(L) = e^{c_1(L)} = \sum_{m=0}^{\infty} \frac{c_1(L)^m}{m!} = 1 + c_1(L) + \frac{c_1(L)^2}{2} + \dots$

$$\left\{\begin{array}{ccc} X & \longrightarrow & BU(n) \\ \downarrow \exists & \nearrow & \\ BU(1) \times \cdots \times BU(1) & & \end{array}\right\}$$

- For a general vector bundle V, the splitting principle says that $V \simeq L_1 \oplus \dots \oplus L_n$ for some line bundles $L_1, \dots, L_n$. Then $ch(V) = e^{c_1(L_1)} + \dots + e^{c_1(L_n)}$.

By construction, $ch(-)$ is a ring homomorphism.

Ex: $ch(-): K(S^2) \longrightarrow H^*(S^2;\mathbb{Q})$ so $ch(x_1) = (\deg 2 \text{ generator in } H^2(S^2;\mathbb{Q}))$.

$L \longmapsto 1 + (\deg 2 \text{ gen.})$

§4. Proof of the Theorem

We are ready to prove the Hopf invariant one theorem.

Proof: If n is odd, then $x^2 = 0 = -x^2$, so assume n is even. Write $n = 2m$.

Consider the cofiber sequence

$$S^{4m-1} \xrightarrow{f} S^{2m} \longrightarrow X \longrightarrow S^{4m} \longrightarrow S^{2m+1}.$$

Apply $K(-)$ and $H^*(-;\mathbb{Q})$ to get a diagram

$$\begin{tikzcd}
 & \mathbb{Z}[x_m]/(x_m^2) \arrow[d, phantom, "\cong" rotate=90] & \mathbb{Z}\oplus\mathbb{Z}x\oplus\mathbb{Z}y \arrow[d, phantom, "\cong" rotate=90] & \mathbb{Z}[x_{2m}]/(x_{2m}^2) \arrow[d, phantom, "\cong" rotate=90] & \\
\cdots & K(S^{2m}) \arrow[l, hook', "0"'] \arrow[d, "ch(-)"] & K(X) \arrow[l, hook', "i^*"'] \arrow[d, "ch(-)"] & K(S^{4m}) \arrow[l, hook', "j^*"'] \arrow[d, "ch(-)"] & \cdots \arrow[l, hook', "0"'] \\
\cdots & H^*(S^{2m};\mathbb{Q}) \arrow[l, hook'] \arrow[d, phantom, "\cong" rotate=90] & H^*(X;\mathbb{Q}) \arrow[l, hook'] \arrow[d, phantom, "\cong" rotate=90] & H^*(S^{4m};\mathbb{Q}) \arrow[l, hook'] \arrow[d, phantom, "\cong" rotate=90] & \arrow[l, hook'] \\
 & \mathbb{Q}[x_m]/(x_m^2) & \mathbb{Q}[x,y]/\begin{pmatrix} y^2=0 \\ x^2=ay \\ xy=0 \end{pmatrix} & \mathbb{Q}[x_{2m}]/(x_{2m}^2) &
\end{tikzcd}$$

The Chern character $ch(-)$ maps x_m to the generator of $H^{2m}(S^{2m};\mathbb{Q})$ for all m, so we denote the generator $x_m \in H^{2m}(S^{2m};\mathbb{Q})$.

Denote
- $y \in \tilde{K}(X)$, $y = j^* x_{2m}$
- $x \in \tilde{K}(X)$, x is a generator such that $i^* x = x_m$.

We then deduce:
- $\underline{y^2 = j^*(x_{2m}^2) = 0}$
- $i^*(x^2) = x_m^2 = 0$, so $\underline{x^2 = ay \text{ for some } a \in \mathbb{Z}}$
- $i^*(xy) = x_m \cdot 0 = 0$, so $\underline{xy = by \text{ for some } b \in \mathbb{Z}}$.
- a is the Hopf invariant, because $ch(x) \in H^{2m}(X;\mathbb{Q})$ is a generator, maps to x_m
  $ch(y) \in H^{4m}(X;\mathbb{Q})$ is the image of x_{2m}.

  thus we assume a is odd.

We also deduce:
- $\underline{\psi^k(x) = k^m x + b_k y}$ for some $b_k \in \mathbb{Z}$
- $\underline{\psi^k(y) = k^{2m} y}$.

Intuition: The Hopf invariant a witnesses how intertwined the classes x and y are. The coefficients b_k in K-theory also witness this.

Now $x^2 = ay$ and $\varphi^2(x) = 2^m x + b_2 y$. By assumption, a is odd.
Since $x^2 \equiv \varphi^2(x) \pmod 2$, this means b_2 is odd.

Now calculate. $\varphi^6(x) = \varphi^3(\varphi^2(x)) = \varphi^3(2^m x + b_2 y) = 2^m(3^m x + b_3 y) + b_2 3^{2m} y$

$\varphi^6(x) = \varphi^2(\varphi^3(x)) = \varphi^2(3^m x + b_3 y) = 3^m(2^m x + b_2 y) + b_3 2^{2m} y$.

$\rightsquigarrow \quad b_3(2^{2m} - 2^m) = b_2(3^{2m} - 3^m) = b_2(3^m - 1)3^m$

Since b_2 is odd, we deduce $2^m \mid 3^m - 1$.

However, (order of 3 mod 2^m) $= 2^{m-2}$ if $m \geq 3$. (ex. order of 3 mod 8 is 2, $3^2 \equiv 1 \bmod 8$)

Thus, $2^{m-2} \mid m$. We easily check that $m = 4$ is the only number ≥ 3 that works.

So $2^m \mid 3^m - 1 \implies m = 1, 2,$ or 4. ∎