Symmetric Powers and the Equivariant Dual Steenrod Algebra

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Abstract

The structure of the Steenrod algebra of stable mod p cohomology operations and its dual $\mathcal{A}_*$ was worked out completely by Milnor - for every prime p, $\mathcal{A}_*$ is a graded-commutative Hopf algebra. However, much of this structure can be alternatively found using the filtration of $H\mathbb{Z}$ coming from the symmetric powers of the sphere spectrum, first studied by Nakaoka, and later by Mitchell-Priddy and others. This filtration not only realizes elements of $\mathcal{A}_*$ explicitly in the homology of certain spaces, but also is the object of the Whitehead Conjecture (proven by Kuhn), and satisfies a duality with the Goodwillie tower of a sphere.

Inspired by the approach of Mitchell-Priddy, we use an equivariant analogue of the symmetric power filtration to try to compute a similar algebra decomposition of $H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p$ in the category of $H\underline{\mathbb{F}}_p$-modules, where $\underline{\mathbb{F}}_p$ is the constant Mackey functor. We focus on the case where $G = C_p$, the cyclic group of order p. In particular, we show that the cofibers in the symmetric filtration of $H\underline{\mathbb{F}}_p$ are Steinberg summands of equivariant classifying spaces, and these cofibers stably split after smashing with $H\underline{\mathbb{F}}_p$. We also explicitly compute equivariant homology decompositions of these spaces when $p = 2$.

Contents

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CHAPTER 1

Introduction

Topologists study topological spaces by means of algebraic invariants and constructions which encode various properties of the spaces - for example, topological K-theory tells us about complex vector bundles, and complex cobordism tells us about cobordism classes of manifolds. These are both examples of *generalized cohomology theories* in the sense of Eilenberg and Steenrod. Generalized cohomology theories themselves form an abelian category with rich structure. If we fix a prime p, we can go one-prime-at-a-time and study the subcategory of p-local theories separately for each prime p. A example of such a theory is $H\mathbb{F}_p$, or *mod p (co)homology* - this can be thought of as a building block in this category.

At the prime $p = 0$, this subcategory is quite nice: in particular, $H\mathbb{Q}$ has no endomorphisms outside of $\mathbb{Q}$ in degree 0, and so there are no nontrivial extensions. This is closely related to the fact that rational homotopy theory can be modeled by algebra ([**33**]), and also to the fact that representations of a finite group split completely into irreducibles in characteristic zero. But at positive primes, $H\mathbb{F}_p$ has lots of nontrivial endomorphisms - its algebra of endomorphisms is called the *Steenrod algebra*. In some sense, this algebra measures the extent to which the p-local world fails to behave rationally.[1]

The Steenrod algebra first appeared in [**37**], [**38**], and [**8**]. A comprehensive computation of the structure of the Steenrod algebra and its dual was done classically

[1]For anyone surprised by this fact, remember that, ironically, even the real world isn't always rational!

by Milnor ([**30**]). For p a prime, we have a decomposition of spectra

$$H\mathbb{F}_p \wedge H\mathbb{F}_p \simeq \bigvee_{\xi} S^{|\xi|} \wedge H\mathbb{F}_p$$

where ξ varies over a collection of monomials which generate (as a module) the dual Steenrod algebra. When $p = 2$, this algebra is isomorphic to $\mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots]$ with $|\xi_k| = 2^k - 1$; when p is odd, it is isomorphic to $\mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes \Lambda[\tau_1, \tau_2, \ldots]$ with $|\xi_k| = 2p^k - 2$ and $|\tau_k| = 2p^k - 1$. (This is actually a *Hopf algebra* - we will discuss its structure in depth in section 2.6.)

The motivating goal of this thesis is to investigate the corresponding version of this story in *equivariant homotopy theory* - i.e., the study of topological spaces with an action of an ambient finite (or profinite/Lie) group G. In this thesis, we focus on the case where $G = C_p$, the cyclic group of order p, where p is the same prime at which we are localizing.[2] That is, we would like a decomposition

$$H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p \simeq \bigvee_{?} ? \wedge H\underline{\mathbb{F}}_p$$

where $H\underline{\mathbb{F}}_p$ is the Eilenberg-Maclane spectrum of the constant *Mackey functor*, and the ?'s are some equivariant analogue of spheres. Hu-Kriz ([**19**], 6.41) calculate, when $p = 2$, an equivariant analogue of the dual Steenrod algebra, roughly (see [**19**] 6.41 for the precise statement)

$$(H\underline{\mathbb{F}}_2)_\star(H\underline{\mathbb{F}}_2) = (H\underline{\mathbb{F}}_2)_\star[\xi_i, \tau_i]/(\tau_i^2 = \tau_{i+1}a_\sigma + \xi_{i+1}u_\sigma)$$

where $\overline{\tau}_i, \overline{\xi}_i$[3] $(i \geq 0)$ correspond to *representation spheres* with dimensions

$$|\overline{\tau}_i| = 2^i + (2^i - 1)\sigma \qquad\qquad |\overline{\xi}_i| = (2^i - 1) + (2^i - 1)\sigma$$

[2]As in representation theory, the interesting behavior occurs when the characteristic of the field divides the order of the group, because unipotence comes into play.

[3]$\overline{\xi}_0 = 1$, as in the ordinary dual Steenrod algebra.

where σ is the sign representation of C_2. They use this computation to study the equivariant spectrum $MU_{\mathbb{R}}$ representing *real* bordism (and the corresponding spectrum $BP_{\mathbb{R}}$). We want to lift this statement to a full decomposition of $H\underline{\mathbb{F}}_2$-modules

$$H\underline{\mathbb{F}}_2 \wedge H\underline{\mathbb{F}}_2 \simeq \bigvee_V S^V \wedge H\underline{\mathbb{F}}_2$$

where V varies over representation spheres formed out of products of the $\overline{\xi}_i$'s and $\overline{\tau}_i$'s, modulo relations.[4]

We utilize an approach which works at all primes. For any finite group G, there is a filtration

$$\mathbb{S}_G \simeq \mathrm{Sp}^1(\mathbb{S}_G) \to \mathrm{Sp}^2(\mathbb{S}_G) \to \mathrm{Sp}^3(\mathbb{S}_G) \to \cdots \to \mathrm{Sp}^\infty(\mathbb{S}_G) \simeq H\underline{\mathbb{Z}}$$

built out of the *symmetric powers* of the *equivariant sphere spectrum* $\mathbb{S}_G$, and converging to $H\underline{\mathbb{Z}}$. Here, we are following after Nakaoka (**[32]**) and many others after him, who studied the nonequivariant version of this filtration and discovered a great deal of structure related to the Steenrod algebra. Some nonequivariant highlights include:

- The n-th cofiber of the filtration is a *suspension spectrum.* **[2]**
- Viewed p-locally, $\mathrm{Sp}^{n-1} \to \mathrm{Sp}^n$ is an equivalence unless n is a power of p. Thus, most of these cofibers are p-locally trivial, and when $n = p^k$, the cofiber admits an explicit description in terms of the *Steinberg representation* of $\mathrm{GL}_k(\mathbb{F}_p)$. Moreover, there is a modification of this filtration which converges to $H\mathbb{F}_p$. **[3]**, **[31]**

[4]I.e., there is a cofiber sequence in the category of $H\underline{\mathbb{F}}_2$-modules, where $E = H\underline{\mathbb{F}}_2 \wedge H\underline{\mathbb{F}}_2$, and I is the ideal of relations

$$I \to E \wedge_{H\underline{\mathbb{F}}_2} E \to E$$

- On mod p cohomology, the attaching maps in this filtration are trivial. The cohomology of the p^k-th cofiber corresponds to admissible sequences of length k in the Steenrod algebra. [**32**], [**31**]
- The attach maps between the cofibers form a long exact sequence on p-local homotopy groups (Whitehead Conjecture, proven in [**21**], [**23**]). Moreover, there is a duality between the cofibers in this filtration and the layers in the Goodwillie tower for a sphere. [**22**] is the most modern work which comprehensively lays out this story.

We follow after their work, and generalize the first three bullet points to the C_2-equivariant setting. Some highlights are:

THEOREM 0.1. *(Chapter 4, 10.1) There is a filtration (arising from symmetric powers)*

$$\mathbb{S}_{C_p} \simeq D_{C_p}(0) \to D_{C_p}(1) \to D_{C_p}(2) \to \cdots \to H\underline{\mathbb{F}}_p$$

where $M_{C_p}(k) := \Sigma^{-k} D_{C_p}(k)/D_{C_p}(k-1)$ *is the Steinberg summand of an equivariant classifying space, namely*

$$\theta_k : M_{C_p}(k) \xrightarrow{\simeq} \epsilon_k B_{C_p}(\mathbb{Z}/p)^k$$

for every k. *Moreover, these isomorphisms relate the respective natural product structures.*

$$\begin{tikzcd}
M_{C_p}(i) \wedge M_{C_p}(j) \arrow[r] \arrow[d] & M_{C_p}(i+j) \arrow[d] \\
\epsilon_i B_{C_p}(\mathbb{Z}/p)^i \wedge \epsilon_j B_{C_p}(\mathbb{Z}/p)^j \arrow[r] & \epsilon_{i+j} B_{C_p}(\mathbb{Z}/p)^{i+j}
\end{tikzcd}$$

THEOREM 0.2. *(Chapter 4, 11.1) There is a graded decomposition*

$$H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p \simeq \bigvee_{k \geq 0} \Sigma^k M_{C_p}(k) \wedge H\underline{\mathbb{F}}_p$$

where $D_{C_p}(n) \wedge H\underline{\mathbb{F}}_p$ is the first $n+1$ summands. The zero-th summand is the unit, and $H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p$ is generated as an $H\underline{\mathbb{F}}_p$-algebra by the zero-th and first summands.

THEOREM 0.3. *(Chapter 5, 4.1) Let $p = 2$. The (desuspension of the) k-th cofiber $\epsilon_k B_{C_2}\Delta_k \wedge H\underline{\mathbb{F}}_2$ can be computed by computing $\epsilon_k S_k$, where S_k is the $RO(G)$-graded ring*

$$S_k := \mathrm{Sym}^*(\mathbb{F}_2^k) \otimes \Lambda^*(\mathbb{F}_2^k) \simeq \mathbb{F}_2[x_1, \ldots, x_k] \otimes \Lambda[s_1, \ldots, s_k]$$

with $\deg(x_i) = \rho_{C_2}$ and $\deg(s_i) = \sigma$ (i.e. the first copy of $\mathbb{F}_2^k$ lies in degree ρ_{C_2} and the second in degree σ), and where GL_k acts in the natural way on each part.

Actually, nearly all of our methods work just as well at odd primes p (and are stated in generality), with the exception of the computations in section 5.3 and 5.4. We hope to fill in this gap with a paper in the near future. In particular, it looks like the generators will not be representation spheres, but to something slightly more exotic.

The thesis is organized as follows. In Chapter 2, we lay out the ideas from the nonequivariant story, focusing on the techniques that we will use. Chapter 3 is dedicated to background material on G-equivariant homotopy theory, with illustrative examples - although sections 3.4 and 3.6 include original computations which we will use later. Chapter 4 is where the real original work begins - here, we use symmetric powers of the G-equivariant sphere to get filtrations for $H\underline{\mathbb{Z}}$ and $H\underline{\mathbb{F}}_p$, and then we compute the geometric fixed points and cofibers in these filtrations. In Chapter 5, we specialize to the prime $p = 2$, and use these previous results to compute the structure of $H\underline{\mathbb{F}}_2 \wedge M_{C_2}(k)$ for every k (where $M_{C_2}(k)$ is the k-th cofiber in the filtration for $H\underline{\mathbb{F}}_2$). For Chapter 3, G is an arbitrary finite group, for Chapter 4 we specialize to $G = C_p$ for p a fixed prime, and in Chapter 5, $G = C_2$. In the later sections of Chapter 5, we discuss unsolved questions which would nicely tie in with the work we have done here, and in Chapter 6, we discuss more open questions and further directions.

Two important proof techniques our paper, which will appear again and again in sections 4 and 5, are

- Prove a result about some C_p-spectra by proving it on both underlying points and geometric fixed points. The underlying points case is often just the nonequivariant analogue of the result, so this reduces many questions to a geometric fixed points computation.
- Prove a result about the symmetric power filtration by first proving it for Sp^p, and then using the product structure $\mathrm{Sp}^{p^i} \wedge \mathrm{Sp}^{p^j} \to \mathrm{Sp}^{p^{i+j}}$ to deduce it for the entire filtration.

CHAPTER 2

Symmetric powers and the Dual Steenrod algebra

In this chapter, we outline the important preliminaries and arguments from the nonequivariant case. Little of the work in this chapter is completely original - we are primarily following after the work of [**31**], but in modern language.

In 2.1, we define the *symmetric power filtration* and more importantly its modulo p variant, which is the central object in our story. In 2.3, we describe the cofibers of both filtrations as Steinberg summands of classifying spaces - thus, 2.2 is devoted to defining the Steinberg idempotent $\epsilon_k \in \mathbb{F}_p[\mathrm{GL}_k(\mathbb{F}_p)]$ and the Bruhat-Tits building (flag complex) of a finite-dimensional vector space over $\mathbb{F}_p$. In section 2.4 and 2.5, we explain how $H\mathbb{F}_p$-modules can be thought of as chain complexes over $\mathbb{F}_p$, and then show that, after smashing with $H\mathbb{F}_p$, the mod p symmetric power filtration splits into graded pieces with an algebra structure.

The remainder of the chapter is more computationally involved, so we focus on the case $p = 2$, although all conceptual results hold at any prime. In 2.6, we explicitly describe the action of GL_k on the homology of the classifying space of $(\mathbb{Z}/2)^k$. 2.7 is a discussion of the Hopf algebra structure on the Steenrod algebra and its dual, and in 2.8, following [**31**], we use these Steenrod operations to compute a basis for the cohomology of the k-th stage of the filtration in terms of *admissible sequences* of length k. In 2.9, we show how the results up to this point can be used algorithmically to recover the ring structure on $H\mathbb{F}_2 \wedge H\mathbb{F}_2$, and in 2.10, we demonstrate how to use a power series argument to recover this ring structure completely. Compiling all of these results, we show

THEOREM 0.4. *There is a graded decomposition*

$$H\mathbb{F}_p \wedge H\mathbb{F}_p \simeq \bigvee_{k\geq 0} \Sigma^k M(k) \wedge H\mathbb{F}_p$$

where $M(k) \simeq \epsilon_k B(\mathbb{Z}/p)^k$. $D(n) \wedge H\mathbb{F}_p$ *is the first* $n+1$ *summands, and there are product maps* $D(i) \wedge D(j) \to D(i+j)$. *The zero-th summand is the unit, and* $H\mathbb{F}_p \wedge H\mathbb{F}_p$ *is generated as an* $H\mathbb{F}_p$*-algebra by the zero-th and first summands. We can explicitly compute the image in* $H\mathbb{F}_2 \wedge H\mathbb{F}_2$ *- for example, when* $p=2$,

$$D(1) \wedge H\mathbb{F}_2 \simeq \mathbb{F}_2\{1, \xi_1, \xi_1^2, \xi_1^3 + \xi_2, \xi^4, \dots\}$$

i.e., the homology generators of $D(1)$ *correspond to the coefficients in the series*

$$(1 + \xi_1 t + \xi_2 t^3 + \xi_3 t^7 + \dots)^{-1} = 1 + \sum_{n>0} (\xi_1 t + \xi_2 t^3 + \xi_3 t^7 + \dots)^n$$

1. The Symmetric Power Filtration

DEFINITION 1.1. *Let* X *be a space. Then its* n*-th* symmetric power *is the space*

$$\mathrm{Sp}^n(X) := X^{\times n}/\Sigma_n = \{x_1 + \dots + x_n : x_i \in X\}$$

This construction is functorial in X.[1]

This can be thought of as formal sums of n (not necessarily distinct) points in X. Thus, we have *sum maps* $\mathrm{Sp}^m(X) \times \mathrm{Sp}^n(X) \to \mathrm{Sp}^{m+n}(X)$. If X has a basepoint, then we can think of the basepoint as the zero element. This gives us *product maps*

$$\mathrm{Sp}^m X \wedge \mathrm{Sp}^n Y \to \mathrm{Sp}^{mn}(X \wedge Y)$$

[1]An alternative way of defining symmetric powers is as follows. Let Fin denote the category whose objects are the finite sets $[n] = \{1, \dots, n\}$ for all $n \geq 1$, and whose morphisms are injections. Let $\mathcal{C}$ be a category which is pointed, closed under colimits, and tensored over sets. Then if $X \in \mathcal{C}$, we have a functor

$$\mathrm{Fin} \xrightarrow{(-)\otimes X} \mathcal{C}$$

and $\mathrm{Sp}^n X$ is the colimit of the diagram formed by the image of the subcategory $\mathrm{Fin}_{\leq n}$ of sets of size at most n. This approach, although less concrete, allows us to directly generalize to spectra.

$$(x_1 + \ldots + x_m) \wedge (y_1 + \ldots + y_n) \mapsto (x_1y_1 + x_1y_2 + \ldots + x_my_n)$$

as well as inclusions

$$X = \mathrm{Sp}^1(X) \hookrightarrow \mathrm{Sp}^2(X) \hookrightarrow \mathrm{Sp}^3(X) \hookrightarrow \cdots \hookrightarrow \mathrm{hocolim}_{n\to\infty}\mathrm{Sp}^n(X) \simeq \mathrm{Sp}^\infty(X)$$

The *infinite symmetric power*, $\mathrm{Sp}^\infty(X)$, is the free topological abelian monoid generated by X. The following theorem of Dold-Thom tells us its homotopy groups.

THEOREM 1.2. **[12]** $\pi_*(\mathrm{Sp}^\infty(X)) \simeq \tilde{H}_*(X;\mathbb{Z})$

In particular, this theorem proves a *group-completion* statement

$$\mathrm{Sp}^\infty X \simeq \mathbb{Z} \otimes X := \{\sum a_i x_i : a_i \in \mathbb{Z}, x_i \in X\}$$

where the basepoint of X is declared to be zero in all sums. As an example, when $X = S^i$

$$\pi_*(\mathrm{Sp}^\infty(S^i)) \simeq \tilde{H}_*(S^i;\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & * = i \\ 0 & * \neq i \end{cases}$$

and therefore, $\mathrm{Sp}^\infty(S^i) \simeq K(\mathbb{Z}, i)$. The inclusion $S^i \hookrightarrow K(\mathbb{Z}, i)$ represents the generator in homotopy.

The symmetric power functors extend to spectra. For example, if $X = \{X_i\}_{i\geq 0}$ is a sequence of spaces equipped with maps $\{f_i : \Sigma X_i \to X_{i+1}\}_{i\geq 0}$, then we have structure maps

$$\Sigma\mathrm{Sp}^n(X_i) \longrightarrow \mathrm{Sp}^n(\Sigma X_i) \xrightarrow{\mathrm{Sp}^n(f_i)} \mathrm{Sp}^n(X_{i+1})$$

which give $\mathrm{Sp}^n(X) = \{\mathrm{Sp}^n(X_i)\}_{i\geq 0}$ the structure of a spectrum. For example, if $\mathbb{S} = \{S^i\}_{i\geq 0}$ is the *sphere spectrum*, then $\mathrm{Sp}^\infty(\mathbb{S}) \simeq H\mathbb{Z}$. This brings us to our main object of study.

DEFINITION 1.3. *Write* $\mathrm{Sp}^n := \mathrm{Sp}^n(\mathbb{S})$. *Then there is a filtration of spectra*

$$\mathrm{Sp}^1 \to \mathrm{Sp}^2 \to \cdots \to \mathrm{Sp}^\infty \simeq H\mathbb{Z}$$

which we call the ***symmetric power filtration****. The sum and product maps on symmetric powers recover the ring structure on* $H\mathbb{Z}$*.*

This filtration exhibits very interesting behavior from the viewpoint of mod p cohomology. The following is a classical result of Nakaoka.

THEOREM 1.4. *(*[**32**]*) If* $p^k \leq n < p^{k+1}$*, then* $H\mathbb{F}_p^*(\mathrm{Sp}^n)$ *has a basis formed by the elements* $\mathrm{St}^I v_0$ *where* St^I *varies over all admissible elements in* $\overline{\mathcal{A}} = H\mathbb{F}_p^*(H\mathbb{Z})$ *of length at most* k*, and* $v_0 \in H\mathbb{Z}^0(\mathrm{Sp}^n)$ *is the fundamental class.*

We will revisit a modern version of this result, and its proof later in this chapter. For now, we note that this result begs the question of whether there is a similar filtration for $H\mathbb{F}_p$ which induces the length filtration on the *full* Steenrod algebra. The answer is yes.

The homomorphism $p : \mathbb{Z} \to \mathbb{Z}$ defines a cofiber sequence $H\mathbb{Z} \to H\mathbb{Z} \to H\mathbb{F}_p$. The induced map on symmetric powers is the *p-replication map* $d : \mathrm{Sp}^n \to \mathrm{Sp}^{pn}$. This arises on the space level from the product map

$$\mathrm{Sp}^p S^0 \wedge \mathrm{Sp}^n X \to \mathrm{Sp}^{pn} X$$

by using p copies of the nontrivial point in S^0. Thus, we obtain a filtration

$$\mathbb{S} \simeq \mathrm{Sp}_p^1 \to \mathrm{Sp}_p^2 \to \mathrm{Sp}_p^3 \to \mathrm{Sp}_p^4 \to \cdots H\mathbb{F}_p$$

where Sp_p^n is obtained from Sp^n by quotienting out the part generated by the images of all $d : \mathrm{Sp}^i \to \mathrm{Sp}^{pi}$ for $i \leq \lfloor n/p \rfloor$. We call this the **mod p symmetric power filtration.**

2. The Steinberg idempotent

The following comes from **[31]**, using **[39]** and **[11]**. Let $k \geq 1$, and let us write $\mathrm{GL}_k = \mathrm{GL}_k(\mathbb{F}_p)$, for brevity. Let $\Sigma_k \subset \mathrm{GL}_k$ be the subgroup of permutations matrices, and let $B_k \subset \mathrm{GL}_k$ be the Borel subgroup of upper triangular matrices. We have elements $\overline{\Sigma}_k, \overline{B}_k \in \mathbb{Z}_{(p)}[\mathrm{GL}_k]$ defined by

$$\overline{\Sigma}_k := \sum_{\sigma \in \Sigma_k} (-1)^\sigma \sigma \qquad\qquad \overline{B}_k := \sum_{b \in B_k} b$$

Then the *Steinberg idempotent* ϵ_k and *conjugate Steinberg idempotent* ϵ_k' are defined by

$$\epsilon_k = \overline{B}_k \overline{\Sigma}_k / [\mathrm{GL}_k : U_k] \qquad\qquad \hat{\epsilon}_k = \overline{\Sigma}_k \overline{B}_k / [\mathrm{GL}_k : U_k]$$

where U_k is a Sylow p-subgroup of GL_k. One can easily check by direct computation that these elements are both idempotent, and therefore if M is any $\mathbb{Z}_{(p)}[\mathrm{GL}_k]$-module, there are inverse isomorphisms

$$\begin{tikzcd} \epsilon_k M \arrow[r, shift left, "\overline{\Sigma}_k"] & \hat{\epsilon}_k M \arrow[l, shift left, "\overline{B}_k"] \end{tikzcd}$$

Later, we will be applying either the Steinberg idempotent or its conjugate, depending on which is more convenient for computation, to homology and cohomology groups, and so we want both of these.

The representation $\mathrm{St}_k := \epsilon_k \mathbb{F}_p[\mathrm{GL}_k]$ is called the *Steinberg representation.* It is *projective* and *self-dual*, and it has dimension $p^{\binom{k}{2}}$, which is the largest power of p dividing $|\mathrm{GL}_k|$. In fact, if we restrict to U_k, then St_k becomes the *regular* representation of U_k!

The Steinberg idempotent is an algebraic object, but it has the following topological avatar. Let $V \simeq \mathbb{F}_p^k$ be a k-dimensional vector space over the field $\mathbb{F}_p$. One can then consider the poset of all *nontrivial* subspaces of V

$$\mathbf{B}_V := \{W \subset V : W \neq 0, V\}$$

This is called the *Bruhat-Tits* building (or just Tits building) of V, and may just be denoted $\mathbf{B}_k$. Automorphisms of V induce self-maps of the poset $\mathbf{B}_V$, and therefore $\mathbf{B}_V$ carries an action of $\mathrm{GL}(V) \simeq \mathrm{GL}_k(\mathbb{F}_p)$.

PROPOSITION 2.1. *([5], Chapter 6) The geometric realization of the nerve of $\mathbf{B}_k$ is homotopy equivalent to a wedge of $p^{\binom{k}{2}}$ copies of S^{k-2}. Its top homology group $H_{k-2}(\mathbf{B}_k; \mathbb{F}_p)$ is St_k, as a representation of GL_k. Equivalently, since ϵ_k is self-dual, its top cohomology group is the same representation.*

The fact that ϵ_k is projective implies that if M is a $\mathbb{Z}_{(p)}[\mathrm{GL}_k]$-module, then it has a decomposition $M \simeq \epsilon_k M \oplus \epsilon_k^{\perp} M$, where $\epsilon_k^{\perp} = 1 - \epsilon_k$ is the orthogonal idempotent. Now suppose that X is a p-local spectrum (see Chapter 3, 5 for the definition) with an action of GL_k. We want to obtain a similar decomposition

$$X \simeq \epsilon_k X \vee \epsilon_k^{\perp} X$$

This decomposition is constructed via a *mapping telescope.* The set of maps $X \to X$ is a $\mathbb{Z}_{(p)}$-module, and therefore we have

$$\epsilon_k^{\perp} X := \mathrm{Tel}(\ X \xrightarrow{\epsilon_k} X \xrightarrow{\epsilon_k} X \xrightarrow{\epsilon_k} \cdots\ )$$

$$\epsilon_k X := \mathrm{Tel}(\ X \xrightarrow{\epsilon_k^{\perp}} X \xrightarrow{\epsilon_k^{\perp}} X \xrightarrow{\epsilon_k^{\perp}} \cdots\ )$$

PROPOSITION 2.2.

$$\epsilon_k X \simeq \Sigma^{1-k}(\mathbf{B}_k^{\diamond} \wedge X)_{h\mathrm{GL}_k}$$

Here, $\mathbf{B}_k^{\diamond}$ is the unreduced suspension of $\mathbf{B}_k$, and therefore has a canonical basepoint given by the point 0.

PROOF. We will construct a map between these which induces an isomorphism on E_*, for any generalized homology E. Pick any nontrivial map $f : S^{k-1} \to \mathbf{B}_k^\diamondsuit$ which is the identity map onto one of the spheres in $\mathbf{B}_k^\diamondsuit$. This yields a map $X \to \Sigma^{1-k}\mathbf{B}_k^\diamondsuit \wedge X$. Then consider the composition

$$\epsilon_k X \to X \to \Sigma^{1-k}\mathbf{B}_k^\diamondsuit \wedge X \to (\Sigma^{1-k}\mathbf{B}_k^\diamondsuit \wedge X)_{h\mathrm{GL}_k}$$

$E_*((\Sigma^{1-k}\mathbf{B}_k^\diamondsuit \wedge X)_{h\mathrm{GL}_k})$ is computed via a spectral sequence with E^2 page

$$H_*(\mathrm{GL}_k; E_*(\Sigma^{1-k}\mathbf{B}_k^\diamondsuit \wedge X)) \simeq H_*(\mathrm{GL}_k; \mathrm{St}_k \otimes E_*X)$$

Since St_k is projective (and therefore flat), this page is concentrated in H_0 and therefore the spectral sequence collapses, with $H_0(\mathrm{GL}_k; \mathrm{St}_k \otimes E_*X) = \epsilon_k(E_*X)$. It is then clear that the composition above, on E_*, gives

$$\epsilon_k(E_*X) \to E_*X \to \mathrm{St}_k \otimes E_*X \to \epsilon_k E_*X$$

which is clearly an isomorphism. □

3. Cofibers in the filtrations

The successive cofibers in the two filtrations described in the previous section admit explicit descriptions.

THEOREM 3.1. *([**2**] Theorem 1.11, [**3**] Theorem 1.1)*

$$\Sigma^\infty(\mathbf{P}_n^\diamondsuit \wedge S^n)_{h\Sigma_n} \simeq \mathrm{Sp}^n/\mathrm{Sp}^{n-1}$$

where $\mathbf{P}_n^\diamondsuit$ is the unreduced suspension of the nerve of the partition poset on n elements. After p-localizing all spectra, $\mathrm{Sp}^n/\mathrm{Sp}^{n-1} \simeq *$ if n is not a power of p, and if $n = p^k$, there is an equivalence

$$\Sigma^\infty(\mathbf{B}_k^\diamondsuit \wedge S^{p^k})_{h\mathrm{Aff}_k} \xrightarrow{\simeq} \mathrm{Sp}^{p^k}/\mathrm{Sp}^{p^k-1}$$

where $\mathbf{B}_k^{\Diamond}$ is the unreduced suspension of the nerve of the Bruhat-Tits building on a k-dimensional vector space over $\mathbb{F}_p$.

Note that $\Sigma^\infty(\mathbf{B}_k^{\Diamond} \wedge S^{p^k})_{h\mathrm{Aff}_k}$ can alternatively be written as $\Sigma^k \epsilon_k B\Delta_k^{\overline{\rho}_k}$, as per our discussion in chapter 2.1. Thus, if we define p-local spectra $L(k) := \mathrm{Sp}^{p^k}/\mathrm{Sp}^{p^{k-1}}$, the theorem above implies that $L(k) \simeq \epsilon_k B\Delta_k^{\overline{\rho}_k}$, where $\Delta_k = (\mathbb{Z}/p)^k$ and $B\Delta_k^{\overline{\rho}_k}$ is the Thom spectrum defined by $\overline{\rho}_k$, the reduced regular representation of Δ_k.

THEOREM 3.2. *([**31**] Theorem A) Define $M(k) := \Sigma^{-k}\mathrm{Sp}_p^{p^k}/\mathrm{Sp}_p^{p^{k-1}}$. Then there is a p-local equivalence*

$$\alpha_k : M(k) \xrightarrow{\simeq} \epsilon_k B\Delta_k$$

and $M(k) \simeq L(k) \vee L(k-1)$. Moreover, the product maps $\mu : M(i) \wedge M(j) \to M(i+j)$ coming from the product maps in the mod p symmetric power filtration, yield a commutative diagram [2]

$$\begin{tikzcd}
M(i) \wedge M(j) \arrow[rr, "\mu"] \arrow[d, "\alpha_i \wedge \alpha_j"] & & M(i+j) \arrow[d, "\alpha_{i+j}"] \\
\epsilon_i B\Delta_i \wedge \epsilon_j B\Delta_j \arrow[r, "\simeq"] & (\epsilon_i \otimes \epsilon_j) B\Delta_{i+j} \arrow[r] & \epsilon_{i+j} B\Delta_{i+j}
\end{tikzcd}$$

These theorems give a procedure for analyzing $H\mathbb{Z} \wedge H\mathbb{F}_p$ and $H\mathbb{F}_p \wedge H\mathbb{F}_p$ by analyzing the cofibers. For example, since we know $B\mathbb{Z}/p \wedge H\mathbb{F}_p \simeq \bigvee_{i \geq 0} \Sigma^i H\mathbb{F}_p$, we have

$$M(k) \wedge H\mathbb{F}_p \simeq \epsilon_k B\Delta_k \wedge H\mathbb{F}_p \simeq \epsilon_k \left(\bigvee_{i_1, \ldots, i_k \geq 0} \Sigma^{i_1 + \ldots + i_k} H\mathbb{F}_p \right)$$

Mitchell-Priddy ([**31**]) perform this computation, and show that the resulting $H\mathbb{F}_p$-module has basis corresponding to the admissible sequences of length k. Moreover,

[2]This second part of the theorem about the product maps is not stated in Mitchell-Priddy's paper, but it follows quite easily, because they construct the projection map $g_k : B\Delta_k \to M(k)$ via the composition

$$B\Delta_k \xrightarrow{\simeq} (B\Delta_1)^{\wedge k} \xrightarrow{g_1^{\wedge k}} M(1)^{\wedge k} \longrightarrow M(k)$$

Note, the fact that ϵ_k is invariant under right-action by the symmetric group Σ_k reflects the fact that the product on the symmetric powers is *commutative*.

the extensions

$$\mathrm{Sp}_p^{p^{k-1}} \to \mathrm{Sp}_p^{p^k} \to \Sigma^k M(k)$$

which attach on the successive cofibers, split after smashing with $H\mathbb{F}_p$.

We will outline this full argument in the final sections of this chapter, as our argument in the equivariant case will follow similar threads.

4. The category of HR-modules

Recall that if R is a commutative ring, then there is an E_∞-ring spectrum HR which represents cohomology with coefficients in R.[3] The following is Theorem IV.2.4 in [**15**]:

THEOREM 4.1. *([**15**], Theorem IV.2.4) Let R be a commutative ring. Then there is an equivalence of categories*

$$C_* : \mathrm{mod}{-}HR \to \mathscr{D}_R$$

given by the cellular chains functor $C_*(-)$. Here, $\mathscr{D}_{HR}$ is the homotopy category of HR-modules localized at the weak equivalences, and $\mathscr{D}_R$ is the derived category of R-modules. In particular, if M is an HR-module, then $\pi_*(M)$ equals the homology of $C_*(M)$.

In particular, this means that if X is a space, then the spectrum $X \wedge HR$ can instead be viewed as the chain complex $R[X] = C_*(X;R)$ in the derived category of R-modules. When $R = \mathbb{F}_p$ (or any field), there is a quasi-isomorphism

$$H_*(X;\mathbb{F}_p) \xrightarrow{\simeq} C_*(X;\mathbb{F}_p)$$

[3]By the Dold-Thom theorem ([**12**]), we have the explicit construction $HR \simeq R \otimes \mathbb{S}$.

This is a result of the fact that there are no nontrivial extensions of $\mathbb{F}_p$-modules. So there is a decomposition

$$X \wedge H\mathbb{F}_p \simeq \bigvee_n (S^n \wedge H\mathbb{F}_p)^{\vee(\dim H_n(X;\mathbb{F}_p))}$$

This reduces the problem of studying the cofibers $M(k) \wedge H\mathbb{F}_p$, to the purely algebraic problem of computing $H_*(M(k);\mathbb{F}_p) = \epsilon_k H_*(B\Delta_k;\mathbb{F}_p)$. From now on, when X is a space (or spectrum), we will interchangeably write $X \wedge H\mathbb{F}_p$ and $\mathbb{F}_p[X]$, depending on which viewpoint is convenient.

5. The Filtration splits on Homology

In this section, we will prove that the cofiber sequence

$$D(k-1) \to D(k) \to \Sigma^k M(k)$$

splits after smashing with $H\mathbb{F}_p$. In fact, we will prove that

THEOREM 5.1. *There is a graded decomposition*

$$H\mathbb{F}_p \wedge H\mathbb{F}_p \simeq \bigvee_{k \geq 0} \Sigma^k M(k) \wedge H\mathbb{F}_p$$

where $D(n) \wedge H\mathbb{F}_p$ is the first $n+1$ summands. The zero-th summand is the unit, and $H\mathbb{F}_p \wedge H\mathbb{F}_p$ is generated as an $H\mathbb{F}_p$-algebra by the zero-th and first summands.[4]

We prove this by first constructing a splitting for $k = 1$

$$\begin{tikzcd} S^0 \wedge H\mathbb{F}_p \arrow[r, shift left] & D(1) \wedge H\mathbb{F}_p \arrow[l, dashed, shift left] \arrow[r] & \Sigma M(1) \wedge H\mathbb{F}_p \end{tikzcd}$$

[4]The product maps

$$(D(i) \wedge H\mathbb{F}_p) \wedge_{H\mathbb{F}_p} (D(j) \wedge H\mathbb{F}_p) \to D(i+j) \wedge H\mathbb{F}_p$$

do not quite respect this decomposition. For example, it is possible to multiply two elements of $M(1) \wedge H\mathbb{F}_p \subset D(1) \wedge H\mathbb{F}_p$ and the result is an element of $D(2) \wedge H\mathbb{F}_p$, but could have nontrivial parts in grading 1 and 2. We will see explicit examples of this in the last two sections of this chapter.

and then extend it to all k by using the product structure. In **[31]**, Mitchell-Priddy construct this splitting explicitly.

PROPOSITION 5.2. *(**[31]**, Proposition 4.4) Let $\lambda : \mathbb{Z}/p \to \mathbb{C}^\times$ be the standard one-dimensional complex representation. Then there is a map*

$$f : (B\mathbb{Z}/p)^{1-\lambda} \to D(1)$$

such that the composite

$$(B\Sigma_p)^{1-\beta} \longrightarrow (B\mathbb{Z}/p)^{1-\alpha} \xrightarrow{tr} (B\mathbb{Z}/p)^{1-\lambda} \xrightarrow{f} D(1)$$

is an equivalence after we quotient out the bottom (negative) cell of $(B\Sigma_p)^{1-\beta}$. Here, α is the complex reduced regular representation of $\mathbb{Z}/p$, β is the complex reduced standard representation of Σ_p, the first map is the inclusion of the Steinberg summand, and the second map is the transfer coming from $S^0 \to S^{\alpha-\lambda}$.

In particular, $H_0(D(1)) \neq 0$, and therefore, $S^0 \wedge H\mathbb{F}_p \to D(1) \wedge H\mathbb{F}_p$ is a monomorphism.

There is a much simpler argument to generate this first splitting than the one Mitchell-Priddy use. Let $\mu : H\mathbb{F}_p \wedge H\mathbb{F}_p \to H\mathbb{F}_p$ be the multiplication map.

$$\begin{tikzcd}
S^0 \wedge H\mathbb{F}_p \arrow[r] \arrow[drr, "\mathrm{Id}"'] & D(1) \wedge H\mathbb{F}_p \arrow[r] \arrow[dr, dashed] & H\mathbb{F}_p \wedge H\mathbb{F}_p \arrow[d, "\mu"] \\
 & & H\mathbb{F}_p
\end{tikzcd}$$

The horizontal composite $S^0 \to H\mathbb{F}_p$ is the unit map, so the composite arrow is the identity. Thus, the dotted arrow gives our splitting map. However, the main reason we reproduce Mitchell-Priddy's argument here is because it explicitly identifies $D(1)$ as a Thom spectrum, i.e.

COROLLARY 5.3. *When $p = 2$, the sequence $S^0 \to D(1) \to \Sigma M(1)$ is the cofiber sequence*

$$S^0 \to (B\mathbb{Z}/2)^{1-L} \to (B\mathbb{Z}/2)^1$$

where the first map comes from the inclusion $S^0 \to (B\mathbb{Z}/2)_+$, and L is the canonical real line bundle on $B\mathbb{Z}/2 \simeq \mathbf{RP}^\infty$.

We will use this identification later when identifying elements of the dual Steenrod algebra in the homology of the filtration (see sections 2.7 and 2.10).

PROOF. *(of Proposition 2.10)* We construct the map $f : (B\mathbb{Z}/p)^{1-\lambda} \to D(1)$ and show that it is an isomorphism on H_0. The conclusion that $H_0(D(1)) \simeq \mathbb{F}_p$ (and therefore that $S^0 \wedge H\mathbb{F}_p \to D(1) \wedge H\mathbb{F}_p$ is a monomorphism) results because, by the Thom isomorphism,

$$\mathbb{F}_p[\Sigma(B\mathbb{Z}/p)^{-\lambda}] \simeq \mathbb{F}_p[S^{-1} \vee S^0 \vee S^1 \vee \cdots]$$

The conclusion that $(B\Sigma_p)^{1-\beta} \to D(1)$ is an isomorphism after quotienting out the bottom cell follows because, by the Thom isomorphism, $(B\Sigma_p)^{1-\beta}$ has homology in degrees $-q+1, 0, 1, q, q+1, 2q, \ldots$ where $q = 2(p-1)$.[5]

For $n \geq 0$, let $S(\mathbb{C}^{n+1}) \simeq S^{2n+1}$ denote the unit sphere in $\mathbb{C}^{n+1}$. This has a free action of $\mathbb{Z}/p$ through λ, and we write L^{2n+1} for the quotient by this action: this is a *lens space*. Clearly, the lens spaces L^{2n+1} form a skeleton for B_p, because as $n \to \infty$, $S(\mathbb{C}^{n+1})$ becomes contractible.

The canonical complex line bundle over $S(\mathbb{C}^{n+1})$ descends to a complex line bundle λ_n over L^{2n+1}. Let $-\lambda_n$ denote its complement: this is a complex vector bundle of dimension n, with total space $\{([x], v) : \langle x, v\rangle = 0\}$ where $x \in S(\mathbb{C}^{n+1})$, $v \in \mathbb{C}^{n+1}$, and $\langle , \rangle$ is the $\mathbb{C}$-valued inner product. Let $D(-\lambda_n)$ denote the unit disc bundle, and

[5]It is a well-known computation in group cohomology that

$$H^*(B\Sigma_p; \mathbb{F}_p) \simeq H^*(B\mathrm{Aff}_1; \mathbb{F}_p) \simeq H^*(B\mathbb{Z}/p; \mathbb{F}_p)^W \simeq \mathbb{F}_p[x, y]^W \simeq \mathbb{F}_p[xy^{p-1}, y^p]$$

where $|x| = 1, |y| = 2$ and $x^2 = 0$. Here, $W \simeq \mathbb{F}_p^\times$ is the Weyl group of $\mathbb{Z}/p \subset \mathrm{Aff}_1$.

let $S(-\lambda_n)$ denote the unit sphere bundle. If L_x is the complex line spanned by x and $|v| \leq 1$, then $L_x + v$ intersects S^{2n+1} in a circle of radius $\sqrt{1-|v|^2}$. Thus, there is a map $\tilde{f}_n : D(-\lambda_n) \to \mathrm{Sp}^p S^{2n+1}$ defined by

$$\tilde{f}_n([x], v) = (v + x\sqrt{1-|v|^2}, v + \zeta x\sqrt{1-|v|^2}, \ldots, v + \zeta^{p-1} z\sqrt{1-|v|^2})$$

where $\zeta = e^{2\pi i/p}$. For points in $S(-\lambda_n)$, this map lands in the diagonal copy of S^{2n+1}, so we have a commutative diagram where the vertical sequences are cofiber sequences

$$\begin{tikzcd}
S(-\lambda_n) \arrow[r] \arrow[d] & S(\mathbb{C}^{n+1}) \simeq S^{2n+1} \arrow[d, "d"] \\
D(-\lambda_n) \arrow[r, "\tilde{f}_n"] \arrow[d] & \mathrm{Sp}^p S(\mathbb{C}^{n+1}) \simeq \mathrm{Sp}^p(S^{2n+1}) \arrow[d] \\
(L^{2n+1})^{-\lambda_n} \arrow[r, "f_n"] & \mathrm{Sp}^p_p(S^{2n+1})
\end{tikzcd}$$

The maps f_n commute with the stabilization maps, and thus we get a map of spectra $f : \Sigma(B\mathbb{Z}/p)^{-\lambda} \to \mathrm{Sp}^p_p \simeq D(1)$. To show this is an isomorphism on H_0, it suffices to show f_n is an isomorphism on H_{2n+1}. Consider the restriction of $\tilde{f}_n$ to the zero section L^{2n+1}. Here, $\tilde{f}_n([x]) = (x, \zeta x, \ldots, \zeta^{p-1}x)$. There is a commutative diagram

$$\begin{tikzcd}
S(\mathbb{C}^{n+1}) \arrow[r, "d"] \arrow[dr, "\pi"'] & S(\mathbb{C}^{n+1})^{\times p} \arrow[r, "1\times\zeta\times\cdots\times\zeta^{p-1}"] & S(\mathbb{C}^{n+1})^{\times p} \arrow[r] & \mathrm{Sp}^p S(\mathbb{C}^{n+1}) \\
& L^{2n+1} \arrow[urr, "\tilde{f}_n"'] & &
\end{tikzcd}$$

Clearly π_* is multiplication by p on H_{2n+1}, as it is a p-fold covering. Since $\zeta^k : S(\mathbb{C}^{n+1}) \to S(\mathbb{C}^{n+1})$ has degree 1, the horizontal composite is also multiplication by p. Thus, $(\tilde{f}_n)_*$ is an isomorphism on H_{2n+1}, as desired. $\square$

Let $t_1 : \mathbb{F}_p[\Sigma^1 M(1)] \to \mathbb{F}_p[D(1)]$ be the splitting of the first stage of the filtration, and define the map $t_k : \mathbb{F}_p[\Sigma^k M(k)] \to \mathbb{F}_p[D(k)]$ by the composite

$$\begin{tikzcd}
\mathbb{F}_p[D(1)^{\wedge k}] \arrow[rr, "\overline{\mu}"] & & \mathbb{F}_p[D(k)] \\
\mathbb{F}_p[(\Sigma\epsilon_1 B\Delta_1)^{\wedge k}] \arrow[u, "t_1^{\otimes k}"] \arrow[rr, shift left, "\mu"] & & \mathbb{F}_p[\Sigma^k \epsilon_k B\Delta_k] \simeq \mathbb{F}_p[\Sigma^k M(k)] \arrow[u, dashed, "t_k"'] \arrow[ll, dashed, shift left, "\iota"]
\end{tikzcd}$$

Here, the bottom horizontal map μ (resp. ι) is projection onto (resp. inclusion of) the Steinberg summand, and the top horizontal map $\overline{\mu}$ comes from the product structure on the symmetric powers.

Proposition 5.4. *t_k is a monomorphism, and therefore splits the sequence*

$$\begin{tikzcd}
\mathbb{F}_p[D(k-1)] \arrow[r] & \mathbb{F}_p[D(k)] \arrow[r, shift right] & \mathbb{F}_p[\Sigma^k M(k)] \arrow[l, dashed, shift right]
\end{tikzcd}$$

Proof.

$$\begin{tikzcd}
\mathbb{F}_p[D(1)^{\wedge k}] \arrow[r, "\overline{\mu}"] \arrow[d, shift left] & \mathbb{F}_p[D(k)] \arrow[d, shift right] \\
\mathbb{F}_p[(\Sigma M(1))^{\wedge k}] \arrow[u, dashed, shift left, "t_1^{\otimes k}"] \arrow[r, shift left, "\mu"] & \mathbb{F}_p[\Sigma^k M(k)] \arrow[u, dashed, shift right, "t_k"'] \arrow[l, dashed, shift left, "\iota"]
\end{tikzcd}$$

The non-dotted arrows clearly form a commutative diagram. We want to show that $t_k = \overline{\mu} \circ t_1^{\otimes k} \circ \iota$ is a monomorphism. It suffices to show that t_k followed by applying the downwards map on the right side of the square, is the identity. This is equivalent to proving that $t_1^{\otimes k} \circ \iota$, followed by the downwards map on the left side of the square and then μ, is the identity. This is obvious because the left side of the square is inclusion of a summand (by the previous corollary) and the bottom of the square is inclusion of the Steinberg summand.[6] $\square$

[6]This graded decomposition *doesn't* respect the product structure! This can be seen in the above diagram: it cannot be the case that $\mu \circ t_1^{\otimes k} = t_k \circ \mu$, because

$$t_k \circ \mu = (\overline{\mu} \circ t_1^{\otimes k} \otimes \iota) \circ \mu = \overline{\mu} \circ t_1^{\otimes k} \otimes (\iota \circ \mu)$$

PROPOSITION 5.5. *The $H\mathbb{F}_p$-module generators of $\mathbb{F}_p[\Sigma M(1)] \subset \mathbb{F}_p[D(1)]$ generate $H\mathbb{F}_p \wedge H\mathbb{F}_p$ as a $H\mathbb{F}_p$-module.*

PROOF. $\mathbb{F}_p[(\Sigma M(1))^{\wedge k}] \to \mathbb{F}_p[\Sigma^k M(k)]$ is projection onto the Steinberg summand, and is therefore surjective. Therefore, the composition $\mathbb{F}_p[D(1)^{\wedge k}] \to \mathbb{F}_p[D(k)] \to \mathbb{F}_p[\Sigma^k M(k)]$ is surjective for every k. It follows that any element of $\mathbb{F}_p[D(k)]$ can be written as a sum of monomials of length $\leq k$ in the module generators of $\mathbb{F}_p[D(1)]$. $\square$

6. Homology of the cofibers (p=2)

Recall that we write $\Delta_k \simeq (\mathbb{Z}/2)^k$. In this section, we will prove the following proposition.

PROPOSITION 6.1. GL_k *acts on the vector space $\langle e_1, \ldots, e_k \rangle$, and its action on the $H\mathbb{F}_2$-algebra*

$$\mathbb{F}_2[B\Delta_k] \simeq \mathbb{F}_2[e_1, \ldots, e_k]$$

is the unique extension of this action (from the degree 1 part) which preserves products.

This will allow us to explicitly analyze the $H\mathbb{F}_2$-module $\mathbb{F}_2[\epsilon_k B\Delta_k] = \epsilon_k \mathbb{F}_2[e_1, \ldots, e_k]$ purely in terms of algebra.

Recall that $B\Delta_k$ has a simplicial structure with n-simplices $(B\Delta_k)_n = \Delta_k^{\times n}$ and structure maps

$$d_0(x_1 \otimes \cdots \otimes x_n) = x_2 \otimes \cdots \otimes x_n$$

$$d_i(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n$$

$$d_n(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_{n-1}$$

$$s_i(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_n$$

---

and the composition $\iota \circ \mu$ is not the identity, but rather is projection onto the Steinberg summand.

We will first analyze $B\Delta_1$. Let Δ_1 be written multiplicatively, as $\Delta = \{1, \tau\}$. It is well-known that $(H\mathbb{F}_2)_*(B\Delta_1)$ is one-dimensional in every degree - we explicitly describe generators on the level of chains.

Proposition 6.2. *The group $(H\mathbb{F}_2)_n(B\Delta_1)$ is generated by the n-simplex $(1+\tau)^{\otimes n}$.*

Proof. Let us write $\alpha = 1+\tau$, so that $\mathbb{F}_2[\Delta_1] = \mathbb{F}_2\{1,\tau\} = \mathbb{F}_2\{1,\alpha\}$ and $\alpha^2 = 0$, i.e. $\{1,\alpha\}$ is an alternative basis for $\mathbb{F}_2[\Delta_1]$. We want to show that $\alpha^{\otimes n}$ generates the homology at $\mathbb{F}_2[\Delta_1^{\times n}]$. It is easy to check that $d(\alpha^{\otimes n}) = 0$. So it suffices to show that this element is not in the image of the differential $\mathbb{F}_2[\Delta_1^{\times(n+1)}] \to \mathbb{F}_2[\Delta_1^{\times n}]$. The image of any element of $\mathbb{F}_2[\Delta_1^{\times(n+1)}]$ is a sum of n-fold tensor products of 1's and α's. The only monomials which have a face equal to $\alpha^{\otimes n}$ are $\alpha^{\otimes(n+1)}$ and $\alpha^{\otimes i} \otimes 1 \otimes \alpha^{\otimes j}$ where $i+j=n$. The first is killed by the differential, so that's out of contention. The second has just two faces equal to $\alpha^{\otimes n}$, and those cancel out in the sum which occurs when we apply the differential. Thus, $\alpha^{\otimes n}$ is not in the image of the differential, and therefore defines a nontrivial element in $H_n(B\Delta_1;\mathbb{F}_2)$. $\square$

We will call the above element e^n - then

$$B\Delta_1 \wedge H\mathbb{F}_2 \simeq H\mathbb{F}_2\{e^0, e^1, e^2, e^3, \ldots\} \simeq \mathbb{F}_2[e]$$

This notation is sensible, because the product map $\mu_1 : B\Delta_1 \wedge B\Delta_1 \to B\Delta_1$ sends $e^i \wedge e^j \mapsto e^{i+j}$ - thus, $H\mathbb{F}_2 \wedge B\Delta_1$ is a ring spectrum with the obvious multiplications.

Proof. *(of 6.1)* By the Kunneth formula,

$$B\Delta_k \wedge H\mathbb{F}_2 \simeq H\mathbb{F}_2\{e_1^{i_1} \cdots e_k^{i_k} : i_1, \ldots, i_k \geq 0\}$$

The product map $\mu_k : B\Delta_k \wedge B\Delta_k \to B\Delta_k$ satisfies

$$\begin{tikzcd} B\Delta_k \wedge B\Delta_k \arrow[r, "\mu_k"] \arrow[d, "\simeq"] & B\Delta_k \\ (B\Delta_1)^{\wedge k} \wedge (B\Delta_1)^{\wedge k} \arrow[r, "\mu_1^{\wedge k}"] & (B\Delta_1)^{\wedge k} \arrow[u, "\simeq"'] \end{tikzcd}$$

so it follows that

$$\mu_k : (e_1^{i_1} \cdots e_k^{i_k}) \wedge (e_1^{j_1} \cdots e_k^{j_k}) \mapsto e_1^{i_1+j_1} \cdots e_k^{i_k+j_k}$$

The action of GL_k respects μ_k, and therefore respects this product structure. It follows that we just need to compute the action of GL_k on e_i, for each i. e_i is represented by the simplex $(1+\tau_i)$, so it suffices for us to show that in $H_1(B\Delta_k)$, if $v, w \in \Delta_k$ are any two vectors, then $1 + vw \sim (1+v) + (1+w)$. But this is clear, because $d(1 \otimes 1 + v \otimes w) = 1 + v + w + vw$.

□

7. The Steenrod Algebra and its Dual (p=2)

For simplicity, we specialize to $p = 2$ in this section. We state some standard results about the structure of the Steenrod algebra $H^*(H\mathbb{F}_2;\mathbb{F}_2)$ and its dual $H_*(H\mathbb{F}_2)$. For these results, we point to the standard reference by Milnor (**[30]**), although there are plenty of other references as well - for example, we like the treatment in **[25]**.

Let $\mathcal{A}^* := H^*(H\mathbb{F}_2;\mathbb{F}_2)$ denote the *Steenrod algebra* of $\mathbb{F}_2$-cohomology operations. This is the graded homotopy of the $H\mathbb{F}_2$-module $\mathrm{Map}(H\mathbb{F}_2, H\mathbb{F}_2)$, with (noncommutative) multiplication coming from composition of cohomology operations. In fact, $\mathcal{A}^*$ is a *Hopf algebra* where the comultiplication is cocommutative.

PROPOSITION 7.1. *$\mathcal{A}^*$ is an $\mathbb{F}_2$-algebra on non-commuting generators* $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^3, \ldots$, *called the* Steenrod squares, *where* $|\mathrm{Sq}^i| = i$. *These generators are subject to the*

Adem relations:

$$\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{\lfloor i/2 \rfloor} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k \qquad (0 < i < 2j)$$

Thus, the Steenrod algebra has an $\mathbb{F}_2$-basis given by monomials $\mathrm{Sq}^I = \mathrm{Sq}^{i_1}\mathrm{Sq}^{i_2}\cdots\mathrm{Sq}^{i_k}$ where $i_1 \geq 2i_2 \geq \ldots \geq 2^{k-1}i_k > 0$ and $k \geq 0$. Such sequences $I = (i_1, \ldots, i_k)$ are called admissible. *(The empty sequence $()$ corresponds to $1 \in \mathcal{A}^*$.)*

Proposition 7.2. *$\mathcal{A}^*$ has a cocommutative comultiplication $\Delta : \mathcal{A}^* \to \mathcal{A}^* \otimes \mathcal{A}^*$ described by*

$$\Delta(\mathrm{Sq}^n) = \sum_{i+j=n} \mathrm{Sq}^i \otimes \mathrm{Sq}^j$$

More generally, if $I = (i_1, \ldots, i_k)$ is an admissible sequence $\Delta(\mathrm{Sq}^I) = \sum_{J+J'} \mathrm{Sq}^J \otimes \mathrm{Sq}^{J'}$, where $J = (j_1, \ldots, j_k)$ and $J' = (j'_1, \ldots, j'_k)$ are arbitrary sequences of nonnegative integers with $j_a + j'_a = i_a$ for $a = 1, \ldots, k$.

Steenrod operations act on $H^*(X; \mathbb{F}_2)$ for any spectrum X. A particularly important example is $X = B\mathbb{Z}/2 \simeq \mathbf{RP}^\infty$. In this case, $H^*(\mathbf{RP}^\infty; \mathbb{F}_2) \simeq \mathbb{F}_2[t]$, and the action of $\mathcal{A}^*$ can be described explicitly:

$$\mathrm{Sq}^i(t^k) = \binom{k}{i} t^{k+i}$$

This is deduced by induction from the case $k = 1$ by using the coproduct Δ on $\mathcal{A}^*$. [7]

By using the comultiplication, we can explicitly describe the action of $\mathcal{A}^*$ on $H^*(B(\mathbb{Z}/2)^k; \mathbb{F}_2) \simeq \mathbb{F}_2[t_1, \ldots, t_k]$. This action commutes with the action of $\mathrm{GL}_k(\mathbb{F}_2)$, an important fact we will use in the next section. This is essentially due to the fact that the Steenrod squares $\mathrm{Sq}^i : H^*X \to H^*X$ are *homomorphisms*, i.e. $\mathrm{Sq}^i(x+y) = \mathrm{Sq}^i x + \mathrm{Sq}^i y$.[8]

[7] If we extend this ring downwards to include the monomial t^{-1}, then we find that $\mathrm{Sq}^i(t^{-1}) = t^{i-1}$ for all i. This computational trick will be used in the next section, following Mitchell-Priddy.

[8] This offers a further explanation of why these operations exist only in positive characteristic, because the equation $(x+y)^p = x^p + y^p$ holds in characteristic p, but no such equation can hold

Within $\mathcal{A}^*$, there are certain important elements $Q_0, Q_1, Q_2, \ldots$ called the *Milnor operations.* These are defined inductively by

$$Q_0 = \mathrm{Sq}^1 \qquad Q_k = \mathrm{Sq}^{2^k} Q_{k-1} + Q_{k-1}\mathrm{Sq}^{2^k} \qquad (k \geq 1)$$

For example, $Q_1 = \mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^3$, $Q_2 = \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^5\mathrm{Sq}^2$, etc. These operations are the *primitives* for the comultiplication operation Δ, that is,

$$\Delta(Q_k) = Q_k \otimes 1 + 1 \otimes Q_k$$

This can be directly checked. The k-th Milnor operation Q_k has degree $2^{k+1} - 1$.

Although the Steenrod operations are useful for computing in cohomology, it is often nicer to consider the *dual Steenrod algebra* $\mathcal{A}_* := H(H\mathbb{F}_2; \mathbb{F}_2)$. This is the $\mathbb{F}_2$-dual of $\mathcal{A}^*$, and therefore it is a *commutative Hopf algebra* (but its comultiplication is not cocommutative). It has an explicit presentation:

Proposition 7.3. *The dual Steenrod algebra is a polynomial algebra*

$$\mathcal{A}_* \simeq \mathbb{F}_2[\xi_1, \xi_2, \xi_3, \ldots] \qquad |\xi_k| = 2^k - 1$$

on commuting generators ξ_k. ξ_k is dual *to Q_{k-1}, and so the ξ_k's are the indecomposables in this algebra - that is, $\{\xi_1, \xi_2, \ldots\}$ are module generators for $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I} = (\xi_1, \xi_2, \ldots)\mathcal{A}_*$ is the augmentation ideal. There is a comultiplication $\psi : \mathcal{A}_* \to \mathcal{A}_* \otimes \mathcal{A}_*$ described by*

$$\psi(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2^i} \otimes \xi_i$$

in characteristic zero. As you might now suspect, there's a reason they're called Steenrod *squares*: there's a way to construct these operations out of the cup-square in cohomology! This is described explicitly in [**5**].

Note that in order to call ξ_1, ξ_2 *indecomposables*, we must have an *augmentation* of $\mathcal{A}_*$. Indeed, we have the product map

$$H\mathbb{F}_2 \wedge H\mathbb{F}_2 \to H\mathbb{F}_2 \simeq S^0 \wedge H\mathbb{F}_2$$

which is an augmentation of $H\mathbb{F}_2 \wedge H\mathbb{F}_2$ in the category of $H\mathbb{F}_2$-modules. If we call its fiber I, then $I/I^2 \simeq \mathbb{F}_2\{\xi_1, \xi_2, \xi_3, \ldots\}$.

8. The Steinberg idempotent and Admissible Sequences (p=2)

In this section, we compute a basis for $\epsilon_k \mathbb{F}_2[e_1, \ldots, e_k]$. So that we can use the action of $\mathcal{A}^*$, we instead perform the calculation on the *dual* $H^*(B\Delta_k) \simeq \mathbb{F}_2[x_1, \ldots, x_k]$, where x_i is dual to e_i.[9] Following **[31]**, we obtain a basis of $\mathbb{F}_2[x_1, \ldots, x_k]\epsilon_k$ corresponding to *admissible sequences of length k*. The reason we include it here is that the structure of their argument is highly relevant to our equivariant generalization.

This proof is by induction, using the fact that $\epsilon_k = \epsilon_{k-1}\overline{A}_k\overline{T}_k$, where T_k is the group of cyclic permutations on $\{1, \ldots, k\}$ and A_k is the subgroup of upper triangular matrices which is zero except on the diagonal and the first row. The crucial tool is Lemma 2.22, which relates the action of the Steenrod algebra and the action of the Steinberg idempotent.

LEMMA 8.1. *Let m be a positive integer with fewer than k ones when written in binary. Then* $\sum\limits_{v \in \mathbb{F}_2^k - 0} v^m = 0$.

PROOF. Let us calculate the coefficient of each monomial $x_1^{i_1} \cdots x_k^{i_k}$ in this sum, where $\sum\limits_{j=1}^{k} i_j = m$. Because the expression is symmetric in the indices, let us assume $i_1 \geq \ldots \geq i_k$. If $i_k = 0$, then the coefficient of $x_1^{i_1} \cdots x_k^{i_k}$ in any v^m is equal to the

[9]Note that in $H^*(B\Delta_1)$, the ring structure comes from the *diagonal* map $B\Delta_1 \to B\Delta_1 \times B\Delta_1$, whereas in $H_*(B\Delta_1)$, the ring structure comes from the *product* map $B\Delta_1 \times B\Delta_1 \to B\Delta_1$. The duality between x^i and e^i is due to the fact that e^i is the nontrivial element of $H_i(B\Delta_1)$ for each i.

coefficient in $(v + x_k)^m$, and the coefficient of this monomial in x_k^m is clearly zero, so the total coefficient in $\sum_{v \in \mathbb{F}_2^k - 0} v^m = 0$ is zero.

Therefore, we may assume $i_k > 0$, and thus $i_j > 0$ for all j. Then, the monomial $x_1^{i_1} \cdots x_k^{i_k}$ appears in only one summand, namely $(x_1 + \ldots + x_k)^m$. Its coefficient is the multinomial coefficient $\binom{m}{i_1, \ldots, i_k} = \frac{m!}{i_1! \cdots i_k!}$. It is well-known that this coefficient is divisible by 2 (and therefore zero) if, when we add $i_1 + \ldots + i_k$ in binary, there are any carries - i.e., if any two of $i_1, \ldots, i_k$ have a one in the same position. Since m was assumed to have fewer than k ones, this addition must have at least one carry, and thus the coefficient of every monomial $x_1^{i_1} \cdots x_k^{i_k}$ is zero. Hence, $\sum_{v \in \mathbb{F}_2^k - 0} v^m = 0$. □

Lemma 8.2. *Write $X_k = x_1 \cdots x_k$, and let I be any sequence of length at most $k-1$. Then*

$$(x_1^{-1} \mathrm{Sq}^I (x_2^{-1} \cdots x_k^{-1})) \epsilon_k = \mathrm{Sq}^I (X_k^{-1})$$

In essence, this lemma tells us that ϵ_k extends the action of A^* to one more variable.

Proof.

$$(x_1^{-1} \mathrm{Sq}^I (x_2 \cdots x_k)^{-1}) \epsilon_k = (x_1^{-1} \mathrm{Sq}^I (x_2 \cdots x_k)^{-1}) \epsilon_{k-1} \overline{A}_k \overline{T}_k = \sum_{i=1}^{k} \sum_{v \in \mathbb{F}_2^k - 0} \pi_i(v) v^{-1} \mathrm{Sq}^I (x_1 \cdots \hat{x}_i^{-1} \cdots x_k)$$

$$= \sum_{v \in \mathbb{F}_2^k - 0} v^{-1} \mathrm{Sq}^I (v X_k^{-1}) = \mathrm{Sq}^I (X_k^{-1})$$

The top two equalities are simply by definition. The third equality holds because $\sum_{i=1}^{k} \pi_i(v) \mathrm{Sq}^I (x_1 \cdots \hat{x}_i^{-1} \cdots x_k) = \mathrm{Sq}^I (v X_k^{-1})$, by linearity. To prove the last equality, we use the comultiplication on $\mathcal{A}^*$. If I is an admissible sequence of length $k-1$, and v is homogeneous of degree one, then

$$\mathrm{Sq}^I (v X_k^{-1}) = v \mathrm{Sq}^I (X_k^{-1}) + v^2 \mathrm{Sq}^{I_1} (X_k^{-1}) + v^4 \mathrm{Sq}^{I_2} (X_k^{-1}) + \ldots + v^{2^{k-1}} \mathrm{Sq}^{I_{k-1}} (X_k^{-1})$$

for some admissible sequences $I_1, \ldots, I_k$. This is because if an admissible sequence of length at most $k-1$ is applied to v, the only possible nonzero results are $v, v^2, v^4, \ldots, v^{2^{k-1}}$. By the previous lemma, when we multiply by v^{-1} and sum over all $v \neq 0$, all terms in the expression above disappear except the first one, which gives $(2^k - 1)\mathrm{Sq}^I(X_k^{-1}) = \mathrm{Sq}^I(X_k^{-1})$. □

THEOREM 8.3. *$\mathbb{F}_2[x_1, \ldots, x_k]\epsilon_k$ has a basis consisting of the (nontrivial) elements*

$$\{\mathrm{Sq}^I(X_k^{-1}) : \ell(I) = k\}$$

PROOF. Let R_k be the $\mathbb{F}_2[x_1, \ldots, x_k]$-module generated by X_k^{-1} and $M_k \subset R_k$ be the submodule spanned by elements $\mathrm{Sq}^I(X_k^{-1})$ as I varies over admissible sequences of any length. We wish to show that $R_k\epsilon_k = M_k$. The lemma implies that $(X_k^{-1})\epsilon_k = X_k^{-1}$, so this proves $M_k \subset R_k\epsilon_k$. To prove the other direction, we will use induction on k. For $k = 1$, this is trivially true. Now suppose we have shown $R_{k-1}\epsilon_{k-1} = M_{k-1}$. Then

$$R_k\epsilon_k = (R_1 \otimes R_{k-1}\epsilon_{k-1})\overline{A}_k\overline{T}_k = (R_1 \otimes M_{k-1})\overline{A}_k\overline{T}_k = (R_1 \otimes M_{k-1})\epsilon_k$$

For any $\mathcal{A}$-module N, $R_1 \otimes N$ is generated by $x_1^{-1} \otimes N$. Thus, it is enough to show that $(x_1^{-1} \otimes M_{k-1})\epsilon_k \subset M_k$, because the action of the Steenrod algebra commutes with the action of GL_k. But this is immediate from the previous lemma.

Now the previous lemma implies that if $\ell(I) \leq k-1$, then $\mathrm{Sq}^I(X_k^{-1})$ does not lie in $\mathbb{F}_2[x_1, \ldots, x_k]$ (i.e., it has some inverses), and it is standard to check that if $\ell(I) \geq k+1$, then $\mathrm{Sq}^I(X_k^{-1}) = 0$. This completes the proof.

□

9. Computing the ring structure (p=2)

Write $1, \Sigma(1), \Sigma(e), \Sigma(e^2), \Sigma(e^3), \ldots$ for the $H\mathbb{F}_2$-module generators of $\mathbb{F}_2[D(1)]$, where e^i is the degree i generator of $\mathbb{F}_2[M(1)]$. Let us compute what these generators and their products correspond to, in terms of the ξ_i's. This algorithm is illustrative, so we include a bit of this explicit computation here.

- Clearly $\Sigma(1)$ is the only element in degree 1, so $\boxed{\xi_1 = \Sigma(1)}$.
- Next, we compute $\xi_1 \cdot \xi_1$. One can check explicitly that $1 \otimes 1 \in \mathbb{F}_2[B\Delta_2]$ is killed by applying the Steinberg idempotent ϵ_2, which implies that $\xi_1^2 \in \mathbb{F}_2[D(2)]$ *maps to zero in* $\mathbb{F}_2[M(2)]$. Therefore (provided that it is nonzero), we know it came from an element of $\mathbb{F}_2[D(1)]$, and thus $\boxed{\Sigma(e) = \xi_1^2}$.
- When we compute $\xi_1 \cdot \xi_1^2$, we see that $1 \otimes e \in \mathbb{F}_2[B\Delta_2]$ is *not* killed by the Steinberg idempotent, and therefore $\xi_1^3 \in \mathbb{F}_2[D(2)]$ maps to a nonzero element in $\mathbb{F}_2[\Sigma^2 M(2)]$. If we call this new element ξ_2, then we have that $\boxed{\Sigma(e^2) = \xi_1^3 + \xi_2}$, and of course by definition, $\boxed{\epsilon_2 \Sigma^2(1 \otimes e) = \xi_2}$.
- One similarly can compute that the next few cells:

$$\boxed{\Sigma(e^3) = \xi_1^4}\ \boxed{\Sigma(e^4) = \xi_1^5 + \xi_1^2\xi_2}\ \boxed{\Sigma(e^5) = \xi_1^6 + \xi_2^2}\ \boxed{\Sigma e^7 = \xi_1^7 + \xi_1^4\xi_2 + \xi_1\xi_2^2 + \xi_3}$$

  in $\mathbb{F}_2[\Sigma M(1)]$,

$$\boxed{\epsilon_2\Sigma^2(1 \otimes e^3 + e \otimes e^2) = \xi_1^2\xi_2}\ \boxed{\epsilon_2\Sigma^2(e \otimes e^3 + e^2 \otimes e^2) = \xi_2^2}$$

  in $\mathbb{F}_2[\Sigma^2 M(2)]$. In particular, the first nontrivial module generator of $\mathbb{F}_2[\Sigma^3 M(3)]$ is the projection from $\mathbb{F}_2[D(3)]$ of the element $\xi_1 \cdot \xi_1^2 \cdot \xi_1^4$. We call this element ξ_3.

The computation above allows us to naturally 'discover' the polynomial generators $\xi_1, \xi_2, \xi_3, \ldots$ of the dual Steenrod algebra - ξ_k is just the first nonzero generator in $\mathbb{F}_2[\Sigma^k M(k)]$.

The reason we have included this computation, is that in Chapter 5, we can follow the same algorithm to discover the generators and relations in the C_2-*equivariant dual Steenrod algebra*, $H\underline{\mathbb{F}}_2 \wedge H\underline{\mathbb{F}}_2$. However, here in the nonequivariant case, there is a slick way to do all of the above computations at once using *formal groups*. We do this in the final section of this chapter.

10. Thom Spectra and Power Series (p=2)

Here, we develop a standard tool, which we will use to identify the image $D(1) \wedge H\mathbb{F}_2 \to H\mathbb{F}_2 \wedge H\mathbb{F}_2$. Recall that $B\mathbb{Z}/2 \simeq \mathbf{RP}^\infty$ has cohomology $H^*(B\mathbb{Z}/2; \mathbb{F}_2) \simeq \mathbb{F}_2[t]$. Let $b_0, b_1, b_2, \ldots$ denote the elements $H_*(B\mathbb{Z}/2; \mathbb{F}_2)$ dual to $1, t, t^2, \ldots$.

Suppose that $\alpha : B\mathbb{Z}/2_+ \to \Sigma^m H\mathbb{F}_2$ represents a degree m cohomology class. Then we have two maps

$$\begin{tikzcd} B\mathbb{Z}/2_+ \arrow[r, "\alpha"] \arrow[dr, "P_\alpha(t)"'] & \Sigma^m H\mathbb{F}_2 \wedge H\mathbb{F}_2 \\ & H\mathbb{F}_2 \wedge \Sigma^m H\mathbb{F}_2 \end{tikzcd}$$

Since $(H\mathbb{F}_2 \wedge H\mathbb{F}_2)^*(B\mathbb{Z}/2) \simeq (H\mathbb{F}_2 \wedge H\mathbb{F}_2)_*[t]$[10], both maps are power series with coefficients in the dual Steenrod algebra. If $\alpha = t^m$, then the top map is just t^m, while the second map is the *power series*

$$P_\alpha(t) = \sum_{i \geq 0} (\alpha_* b_i) t^i$$

LEMMA 10.1. *If $\alpha : B\mathbb{Z}/2_+ \to \Sigma^m H\mathbb{F}_2$ and $\beta : B\mathbb{Z}/2_+ \to \Sigma^n H\mathbb{F}_2$ are cohomology classes, let $\alpha \cup \beta : B\mathbb{Z}/2_+ \to \Sigma^{m+n} H\mathbb{F}_2$ denote their cup product. Then*

$$P_{\alpha \cup \beta}(t) = P_\alpha(t) P_\beta(t)$$

[10] This relies on the fact that when $E = H\mathbb{F}_2$ or $E = H\mathbb{F}_2 \wedge H\mathbb{F}_2$, we have an *orientation* $t \in E^*(\mathbf{RP}^\infty)$ such that there is a commutative diagram

$$\begin{tikzcd} \mathbf{RP}^\infty_+ \arrow[r, "t"] & \Sigma^1 E \\ S^1 \arrow[u] \arrow[ur, "\text{unit}"'] & \end{tikzcd}$$

When $E = H\mathbb{F}_2 \wedge H\mathbb{F}_2$, this element is chosen to come from the first coordinate, which is why α and $P_\alpha(t)$ look different.

PROOF. We wish to show that

$$(\alpha \cup \beta)_*(b_k) = \sum_{i+j=k} (\alpha_* b_i) \cdot (\beta_* b_j)$$

This holds because in the following commutative diagram

$$\begin{tikzcd} B\mathbb{Z}/2_+ \arrow[r, "\alpha\cup\beta"] \arrow[d, "\Delta"] & \Sigma^{m+n} H\mathbb{F}_2 \\ (B\mathbb{Z}/2 \times B\mathbb{Z}/2)_+ \arrow[r, "\alpha\wedge\beta"] & \Sigma^m H\mathbb{F}_2 \wedge \Sigma^n H\mathbb{F}_2 \arrow[u] \end{tikzcd}$$

the map Δ on the left sends $b_k \mapsto \sum_{i+j=k} b_i \wedge b_j$. □

Now suppose that V is a virtual real representation of $\mathbb{Z}/2$ of dimension n. Then the associated *Thom space* $(B\mathbb{Z}/2)^V$ has a *Thom class* $u_V : (B\mathbb{Z}/2)^V \to \Sigma^n H\mathbb{F}_2$. In particular, the composite of this map with the 0-section map

$$B\mathbb{Z}/2 \to (B\mathbb{Z}/2)^V \xrightarrow{u_V} \Sigma^n H\mathbb{F}_2$$

represents the *Euler class* $e(V)$ of V. Let $P_V(t)$ denote the power series $P_{e(V)}(t)$.

PROPOSITION 10.2. *$P_{V\oplus W}(t) = P_V(t) \cdot P_W(t)$, i.e. the product of the power series.*

PROOF. This is immediate from the fact that $e(V \oplus W) = e(V) \cup e(W)$. This can be deduced from the multiplicativity of the Thom class, i.e. the square on the right in the commutative diagram below.

$$\begin{tikzcd} B\mathbb{Z}/2_+ \arrow[r] \arrow[d, "\Delta"'] & (B\mathbb{Z}/2)^{V\oplus W} \arrow[r, "u_{V\oplus W}"] \arrow[d, "\Delta"'] & \Sigma^{m+n} H\mathbb{F}_2 \\ B\mathbb{Z}/2_+ \wedge B\mathbb{Z}/2_+ \arrow[r] & (B\mathbb{Z}/2)^V \wedge (B\mathbb{Z}/2)^W \arrow[r, "u_V \wedge u_W"] & \Sigma^n H\mathbb{F}_2 \wedge \Sigma^m H\mathbb{F}_2 \arrow[u, "\mu"'] \end{tikzcd}$$

□

For example, when V is the *trivial* bundle $\mathbf{n}$, $P_{\mathbf{n}}(t) = 1 \cdot t^n$ where $1 \in H_0 H\mathbb{F}_2$.

Now let L denote the standard sign representation $\mathbb{Z}/2 \to \mathbb{R}^{\times}$. The associated real line bundle is equal to the *canonical line bundle* over $\mathbf{RP}^{\infty}$. We wish to compute the power series $P_{L-1}(t)$.

Proposition 10.3. *$P_L(t_1 + t_2) = P_L(t_1) + P_L(t_2)$, and therefore, $P_{L-1}(t)$ has coefficient zero for each t^i except when i is of the form $i = 2^n - 1$. In fact,*

$$P_{L-1}(t) = \sum_{n \geq 0} \xi_n t^{2^n - 1}$$

Proof. We will just prove that P_L is additive. $e(L) = t$, because t is the restriction to $B\mathbb{Z}/2$ of the Thom class of L. Now it is well-known that the product map

$$\times : B\mathbb{Z}/2 \times B\mathbb{Z}/2 \to B\mathbb{Z}/2$$

induces the *additive formal group law* on $H^*(B\mathbb{Z}/2)$,

$$\begin{tikzcd}
(B\mathbb{Z}/2 \times B\mathbb{Z}/2)_+ \arrow[d, "\times"'] \arrow[dr, "t_1+t_2"] & \\
B\mathbb{Z}/2_+ \arrow[r, "t"] & \Sigma^1 H\mathbb{F}_2
\end{tikzcd}$$

Thus, after smashing with $H\mathbb{F}_2$, we get a diagram

$$\begin{tikzcd}
(B\mathbb{Z}/2 \times B\mathbb{Z}/2)_+ \wedge H\mathbb{F}_2 \arrow[d, "\times"'] \arrow[dr, "t_1+t_2"] & \\
B\mathbb{Z}/2_+ \wedge H\mathbb{F}_2 \arrow[r, "t"] & \Sigma^1 H\mathbb{F}_2 \wedge H\mathbb{F}_2
\end{tikzcd}$$

The map $t_1 + t_2$ shown comes from the sum of the composites

$$\begin{tikzcd}
(B\mathbb{Z}/2 \times B\mathbb{Z}/2)_+ \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & B\mathbb{Z}/2_+ \arrow[r, "t"] & \Sigma^1 H\mathbb{F}_2
\end{tikzcd}$$

which implies $P_L(t_1 + t_2) = P_L(t_1) + P_L(t_2)$. $\square$

Recall, by ([**31**] Proposition 4.4), that $\mathrm{Sp}_2^2 =: D(1)$ is the Thom spectrum $(B\mathbb{Z}/2)^{1-L}$. The inclusion $D(1) \to H\mathbb{F}_2$ represents the nontrivial element of $H^0(D(1);\mathbb{F}_2)$, i.e. the Thom class of this bundle. Therefore, we can calculate the effect of the inclusion $D(1) \to H\mathbb{F}_2$ on homology.

PROPOSITION 10.4. *The images of the generators of $H_*D(1)$ under the map $H_*D(1) \to H_*H\mathbb{F}_2 = \mathcal{A}_*$ are the coefficients of $t^0, t^1, t^2, \ldots$ in the power series*

$$(1 + \xi_1 t + \xi_2 t^3 + \xi_3 t^7 + \ldots)^{-1} = 1 + \sum_{n>0}(\xi_1 t + \xi_2 t^3 + \xi_3 t^7 + \ldots)^n$$

$$= 1 + \xi_1 t + \xi_1^2 t^2 + (\xi_1^3 + \xi_2)t^3 + \xi_1^4 t^4 + (\xi_1^5 + \xi_1\xi_2^2)t^5 + \ldots$$

PROOF. This follows immediately from the last two propositions. $\square$

COROLLARY 10.5. *The image of $D(k) \wedge H\mathbb{F}_2 \hookrightarrow H\mathbb{F}_2 \wedge H\mathbb{F}_2$ is the linear span of all monomials of length at most k in the generators $\xi_1, \xi_1^2, \xi_1^3 + \xi_2, \xi_1^4, \xi_1^5 + \xi_1\xi_2^2, \ldots$.*

We finish this section with an alternative interpretation of this Thom spectrum $(B\mathbb{Z}/2)^{1-L}$, as the cofiber of the *stable transfer* map $B\mathbb{Z}/2_+ \to S^0$. This map arises as follows. Recall that there is a cofiber sequence

$$\cdots \to (C_2)_+ \to S^0 \to S^\sigma \to \cdots$$

Apply Spanier-Whitehead duality to obtain the (stable) cofiber sequence

$$\cdots \to S^{-\sigma} \to S^0 \to (C_2)_+ \to S^{1-\sigma} \to \cdots$$

The middle map $\Sigma^\infty S^0 \to \Sigma^\infty (C_2)_+$ is a transfer map (see section 3.2, for example). If we now take homotopy orbits under C_2, the right three terms give us

$$\cdots \to B\mathbb{Z}/2_+ \to S^0 \to (B\mathbb{Z}/2)^{1-L} \to \cdots$$

The transfer map $B\mathbb{Z}/2_+ \to S^0$ is precisely the first attaching map $M(1) \to S^0$.

CHAPTER 3

Equivariant Homotopy Theory

This chapter is primarily devoted to background material on G-equivariant stable homotopy theory. As this is a rich theory with plenty of standard references in the literature, we only focus on the highlights we need. In 3.1, we define orthogonal G-spectra, and some of the standard constructions on them. In 3.2, we define Mackey functors, and how each defines an Eilenberg-Maclane spectrum. In 3.3, we discuss how if $\underline{R}$ is a Tambara functor, then $H\underline{R}$-modules can be viewed as chain complexes over $\underline{R}$. In 3.4, we discuss equivariant homotopy orbits, and give an explicit cell structure for the universal Λ-free space $E_G\Lambda$. In 3.5, we briefly mention the definition of p-localization of G-spaces - this is because we want to consider all spectra to be p-localized. In 3.6, we prove a lemma we will need about the Steinberg idempotent in the equivariant setting.

1. Representation Spheres

Let V be a finite-dimensional orthogonal (real) representation of G (henceforth referred to as just *representations*). Then we let S^V denote the one-point compactification of V. This is a pointed G-space whose H-fixed points, for any $H \subset G$, are given by S^{V^H}.

The spheres $S^n = S^{\mathbb{R}^n}$ form the building blocks for ordinary stable homotopy theory. The passage from ordinary spaces to spectra is effectively done by formally inverting the suspension functors Σ^n by introducing negative spheres S^{-n}. This turns the adjunction between the functors Σ and Ω into an inverse relationship. Meanwhile, in the equivariant setting, these *representation spheres* are the fundamental objects.

For example, when X is a pointed G-space we can define

$$\Sigma^V X := S^V \wedge X \qquad ; \qquad \Omega^V X := \mathrm{Map}_*(S^V, X)$$

$$\underline{\pi}_V^{(-)}(X) := \mathrm{Map}_*(S^V, X)^{(-)}$$

this last object being a *Mackey functor* (to be discussed in the next section). In particular, if we wish to construct a nicely-behaved category of G-spectra with Spanier-Whitehead duals, Thom isomorphism, etc. then we need to stabilize with respect to all of these representation spheres, not just the S^n's. Thus, we get the following definition

DEFINITION 1.1. *An* equivariant orthogonal spectrum *(or G-*spectrum*) X consists of a collection of G-spaces $\{X_V\}$ indexed over representation spheres V, along with compatible structure maps*

$$S^{V-W} \wedge X_W \to X_V$$

whenever $W \subset V$ ($V - W$ denotes the orthogonal complement of W in V). We write $\mathbb{S}_G$ to mean the equivariant sphere spectrum*, i.e. the G-spectrum given by the representation spheres with the obvious structure maps.*

A G-space can be viewed as an H-space for any $H \subset G$. G-spaces also have *fixed points* under various subgroups. We describe analogous constructions on G-spectra.

DEFINITION 1.2. *The* geometric fixed points *of X are given by the nonequivariant spectrum*

$$(\Phi^G X)_n = X(n\rho_G)^G$$

where ρ_G is the regular representation of G. This construction is monoidal up to weak equivalence, i.e. $\Phi^G(X \wedge Y) \simeq \Phi^G X \wedge \Phi^G Y$, and if X is a pointed G-space, then

$$\Phi^G(\Sigma^\infty X) \simeq \Sigma^\infty X^G$$

DEFINITION 1.3. *Let X be a G-spectrum. Then for each $H \subset G$, we have an H-spectrum $i_H^*(X)$ obtained by restricting all actions of G to the subgroup H. For any subgroup $H \subset G$, we define $\Phi^H X = \Phi^H(i_H^* X)$.*

A standard property of G-spaces is that a map $f : X \to Y$ is an equivalence if and only if it is an equivalence on fixed points $f^H : X^H \xrightarrow{\simeq} Y^H$ for every subgroup $H \subset G$. The analogous property for geometric fixed points of G-spectra, which we will use, also holds.

PROPOSITION 1.4. *([**18**], Proposition 2.52) If $f : X \to Y$ is a map of G-spectra such that $\Phi^H f : \Phi^H X \to \Phi^H Y$ is an equivalence for every subgroup $H \subset G$, then f is an equivalence.*

We use $\mathscr{S}^G$ to denote the category of G-spectra, and have adjoint functors

$$\Sigma^\infty : \mathrm{Top}_*^G \rightleftarrows \mathscr{S}^G : \Omega^\infty$$

where Top_*^G is the category of pointed G-spaces. See [**26**] for a complete axiomatic introduction. Two important properties we get from this construction:

(1) If S and T are finite G-sets with a map $S \to T$, we have wrong-way maps $\Sigma_+^\infty T \to \Sigma_+^\infty S$: these are called *transfer maps*. For example, when $S \to T$ is an inclusion, the composition $\Sigma_+^\infty S \to \Sigma_+^\infty T \to \Sigma_+^\infty S$ is multiplication by $|T|/|S|$. These will be motivated in the next section.

(2) Homotopy classes of maps between G-spectra can be added, as in the nonequivariant case.[1] In fact, the homotopy classes of maps between two G-spectra form a module over the *Burnside ring*.

Example: Let us finish this section with a discussion of representation spheres when $G = C_2$, the cyclic group of order 2. Here, there are two irreducible real

[1]In the nonequivariant case, this structure appears because *infinite loopspaces* $\Omega^\infty X$ have a composition structure which is commutative up to coherent homotopy, also called an E_∞-ring structre.

representations, both of dimension one: the *trivial* representation (denoted by 1), and the *sign* representation (denoted by σ). S^1 has a cell decomposition with a trivial 0-cell and a trivial 1-cell, with the attaching map $S^0 \to *$ being the only one possible. S^σ has a cell decomposition with two trivial 0-cells and one *free* 1-cell, where the attaching map $(C_2)_+ \to S^0$ sending both points in the free cell to the non-basepoint. In other words, we have cofiber sequences

$$S^0 \to * \to S^1 \qquad ; \qquad (C_2)_+ \to S^0 \to S^\sigma$$

In general, $S^{(n+1)\sigma}$ can be built from $S^{n\sigma}$ via a cofiber sequence which attaches on one more free cell in degree $n+1$:

$$S^{n+1} \wedge (C_2)_+ \to S^{n\sigma} \to S^{(n+1)\sigma}$$

2. Mackey Functors and Eilenberg-Maclane spectra

Suppose that M is a left G-module. Then M can be viewed as a monoidal functor

$$M : (G\mathrm{Set}^{\mathrm{op}}, \sqcup) \to (\mathrm{Ab}, \oplus)$$

whose value on a transitive G-set G/H is the fixed point group M^H. In particular, for every inclusion of subgroups $K \subset H$, we have a quotient map $G/K \to G/H$, and correspondingly have a *restriction* map $M^H \to M^K$. However, the abelian group structure gives *transfer maps* going the other way

$$M^K \to M^H$$

$$m \mapsto \sum_{g \in H/K} gm$$

Thus, M can be viewed as a functor from the *Burnside category* to abelian groups. Such functors in general (which don't always arise from G-modules) are called *Mackey functors*.

Definition 2.1. *Let* Burn_G *denote the monoidal category of finite* G*-sets under coproduct where* $\mathrm{Mor}(S,T)$ *is the set of equivalences classes of* correspondences

$$S \leftarrow S' \rightarrow T$$

where S' *is a finite* G*-set. Composition of morphisms is given by pullback of correspondences. Then a functor*

$$M : \mathrm{Burn}_G \rightarrow (\mathrm{Ab}, \oplus)$$

is called a Mackey functor*. (*Burn_G *is just the full subcategory of* $\mathscr{S}^G$ *given by the* $\Sigma^\infty T_+$*, as* T *ranges over finite* G*-sets.)*

The category of Mackey functors (G fixed) clearly forms an abelian category. If M is a G-module, we sometimes denote its associated fixed points Mackey functor by $\underline{M}$, to avoid confusion. (More generally, when there might be confusion, we put an underline to indicate we are talking about a Mackey functor.) If $M : G\mathrm{Set} \rightarrow \mathrm{Ab}$, then we have a natural left Kan extension

$$\begin{tikzcd} G\mathrm{Set} \arrow[r, "M(-)"] \arrow[d] & \mathrm{Ch}_{\mathrm{Ab}} \\ G\mathrm{sSet} \arrow[ur, dashed, "C_*(-;M)"'] & \end{tikzcd}$$

which allow us to construct M-chains on any G-space. Similarly, if $M : G\mathrm{Set}^{\mathrm{op}} \rightarrow \mathrm{Ab}$, then we can construct M-cochains. A Mackey functor has both simultaneously.

In the non-equivariant setting, abelian groups A define homology and cohomology theories, and we represent these on the spectrum level by *Eilenberg-Maclane spectra* HA. We have something similar for Mackey functors.

Theorem 2.2. *(**[27]**) Let* $\underline{M}$ *be a Mackey functor. Then there is a* G*-spectrum* $H\underline{M}$ *with the property that* $\underline{\pi}_0(H\underline{M}) = \underline{M}$ *and* $\underline{\pi}_n(H\underline{M}) = 0$ *for* $n \neq 0$*.* $\underline{M}$*-homology* *and $\underline{M}$-cohomology are given respectively by*

$$H_*^{(-)}(X;\underline{M}) = \underline{\pi}_*^{(-)}(X \wedge H\underline{M})$$

$$(H^*)^{(-)}(X;\underline{M}) = \underline{\pi}_*^{(-)}Map_*(X, H\underline{M})$$

and these are both $RO(G)$-graded, i.e. graded on representations of G. (See [**27**] *for a more thorough introduction.)*[2]

Just as a Mackey functor $\underline{M}$ defines an equivariant Eilenberg-Maclane spectrum $H\underline{M}$, there's an analogous, richer structure which defines an equivariant Eilenberg-Maclane *ring spectrum*. These are *Tambara functors*: functors from the Burnside category to the category of rings, and in addition to restriction maps and the additive transfers given above, there are multiplicative *norm* maps, along with various conditions on how these maps interact. See [**41**] for the original paper, or [**40**] for a more recent introduction. The primary fact is that

Theorem 2.3. *(*[**43**]*) If $\underline{R}$ is a Tambara functor, then $H\underline{R}$ has the structure of an equivariant E_∞-ring spectrum.*

For example, any G-module with a ring structure is a Tambara functor. For our purposes, these are the only Tambara functors we will need to consider.

3. The category of H$\underline{\text{R}}$-modules

Our fundamental tool for understanding $X \wedge H\underline{\mathbb{F}}_p$ when X is a G-space, is the following theorem of Schwede-Shipley that is analogous to (Chapter 2, 4.1).

Theorem 3.1. *(*[**36**]*, Theorem 5.1.6) There is a Quillen equivalence of categories*

$$\text{mod}-H\underline{R} \stackrel{\simeq}{\to} \text{Ch}_{\underline{R}}$$

[2] In fact, just as $HA \simeq A \otimes \mathbb{S}$ in the nonequivariant case, we similarly have a concrete construction due to dos Santos ([**13**])

$$H\underline{M} \simeq \underline{M} \otimes \mathbb{S}_G$$

We will use this construction in the next chapter to get the *equivariant symmetric power filtration.*

between the homotopy category of $H\underline{R}$-modules and the homotopy category of chain complexes of $\underline{R}$-module Mackey functors.

If X is a G-space, then the image of $X \wedge H\underline{R}$ under this equivalence can be described quite explicitly. First, suppose that T is a G-set. Then we can define the *permutation Mackey functor* $\underline{R}_T$ by

$$\underline{R}_T(S) := \underline{R}(T \times S)$$

This is an $\underline{R}$-module in the obvious way. Then $\Sigma^\infty_+ T \wedge H\underline{R} = H\underline{R}_T$. This construction has a natural left Kan extension to G-simplicial sets X.

$$\begin{tikzcd} G\mathrm{Set} \arrow[r, "\underline{R}_{(-)}"] \arrow[d] & \mathrm{Ch}_{\underline{R}} \\ G\text{-sSet} \arrow[ur, dashed, "{\underline{R}[-]}"'] & \end{tikzcd}$$

Concretely, this is done by just applying $\underline{R}_{(-)}$ to the simplices at every level of X. Then $\underline{R}[X]$ is the chain complex of $\underline{R}$-module Mackey functors associated to $X \wedge H\underline{R}$. We will interchangeably use either notation, $\underline{R}[X]$ or $X \wedge H\underline{R}$, depending on which is convenient.

As a concrete example, if $G = C_2$, and $\underline{R}$ is the constant Mackey functor $\underline{\mathbb{Z}}$ (arising from the trivial module $\mathbb{Z}$), we have

$$S^{i\sigma} \wedge H\underline{\mathbb{Z}} \simeq \underline{\mathbb{Z}}[S^{i\sigma}] \simeq (\underbrace{\mathbb{Z}[C_2] \to \cdots \to \mathbb{Z}[C_2]}_{i \text{ copies}} \to \mathbb{Z})$$

4. Equivariant Homotopy Orbits

Let Λ be any finite group. Then $E_G\Lambda$ is the $(G\times\Lambda)$-space with the property that, for any subgroup $\Gamma \subseteq G\times\Lambda$,

$$(E_G\Lambda)^\Gamma \simeq \begin{cases} * \text{ if } \Gamma \in \mathcal{F} \\ \emptyset \text{ if } \Gamma \notin \mathcal{F} \end{cases}$$

where $\mathcal{F}$ is the collection of *graph subgroups* - i.e., $\Gamma\in\mathcal{F}$ if and only if $\Gamma\cap\Lambda=\{1\}$. Note that $\mathcal{F}$ is closed under conjugation and closed under taking subgroups, so this is a sensible definition. The goal of this section is to construct a simple explicit simplicial model for $E_G\Lambda$ when $G=C_p$, i.e. 4.3. We will use this model in Chapter 5 to analyze the equivariant classifying space $B_{C_2}(\mathbb{Z}/2)^k$.

It is easy to check that if Λ_1 and Λ_2 are two groups, then $E_G\Lambda_1\times E_G\Lambda_2\simeq E_G(\Lambda_1\times\Lambda_2)$ - this is a consequence of the fact that a subgroup of $G\times(\Lambda_1\times\Lambda_2)$ is a graph subgroup if and only if its projections onto $G\times\Lambda_1$ and $G\times\Lambda_2$ are both graph subgroups. If we forget the action of G and consider $E_G\Lambda$ as a Λ-space, we get $E\Lambda$. $E_G\Lambda$ is the universal Λ-free $(G\times\Lambda)$-space - thus, if X is a $(G\times\Lambda)$-space, we define the *genuine homotopy orbits* by

$$(X)_{h_G\Lambda} := X\times_\Lambda E_G\Lambda$$

We call $B_G\Lambda=(*)_{h_G\Lambda}$ the *equivariant classifying space*, and clearly $B_G\Lambda_1\times B_G\Lambda_2\simeq B_G(\Lambda_1\times\Lambda_2)$.

We now construct an explicit simplicial model for $E_G\Lambda$.

Definition 4.1. *The $(G\times\Lambda)$-simplicial set $E_G(\Lambda)$ has n-simplices*

$$E_G(\Lambda)_n = \left(\coprod_\Gamma (G\times\Lambda)/\Gamma\right)^{\times(n+1)}$$

with the obvious face and degeneracy maps. Here, Γ varies over all conjugacy classes of nontrivial graph subgroups of $G \times \Lambda$.

Proposition 4.2. *$E_G(\Lambda) \simeq E_G\Lambda$, and when $G = C_p$, $E_{C_p}(\Lambda_1) \times E_{C_p}(\Lambda_2) = E_{C_p}(\Lambda_1 \times \Lambda_2)$.*

Proof. We check that $E_G(\Lambda)$ has the correct fixed points under all subgroups $\Theta \subset G \times \Lambda$. The isotropy group of any element of $(G \times \Lambda)/\Gamma$ is conjugate to Γ. Therefore, if Θ is not contained in a conjugate of a graph subgroup (i.e., Θ is not itself a graph subgroup), then $E_G(\Lambda)^\Theta = \emptyset$. On the other hand, if Θ is a graph subgroup, then $E_G(\Lambda)^\Theta$ is the bar construction on the nonempty set $\coprod_\Gamma((G \times \Lambda)/\Gamma)^\Theta$, and is therefore contractible.

The second part of the proposition follows from the fact that any nontrivial graph subgroup of $C_p \times \Lambda$ corresponds to a map $C_p \to \Lambda$, and conjugacy classes of maps $C_p \to \Lambda_1 \times \Lambda_2$ correspond to pairs of conjugacy classes of maps $C_p \to \Lambda_1$ and $C_p \to \Lambda_2$. $\square$

When $G = C_p$, there is a second, smaller simplicial model sitting inside $E_{C_p}(\Lambda)$.

Definition 4.3. *For each nontrivial graph subgroup Γ, let $K(\Lambda)_\Gamma \subset E_{C_p}(\Lambda)$ denote the simplicial set whose n-simplices are*

$$(K(\Lambda)_\Gamma)_n = \Lambda_0^{(n+1)} \sqcup (\Lambda_0^n \times \Lambda_\Gamma) \sqcup \cdots \sqcup (\Lambda_0 \times \Lambda_\Gamma^n) \sqcup \Lambda_\Gamma^{(n+1)}$$

where $\Lambda_\Gamma := (C_p \times \Lambda)/\Gamma$, and $\Lambda_0 := (C_p \times \Lambda)/C_p$. Let $K(\Lambda) = \bigcup_\Gamma K(\Lambda)_\Gamma$.

Proposition 4.4. *$K(\Lambda) \hookrightarrow E(\Lambda)$ is an equivalence, and thus the inclusion $K(\Lambda_1 \times \Lambda_2) \to K(\Lambda_1) \times K(\Lambda_2)$ is an equivalence.*

Proof. It is equivalent to prove that $K(\Lambda) \simeq E_{C_p}\Lambda$. We check that $K(\Lambda)$ has the correct fixed points under all subgroups $\Theta \subset C_p \times \Lambda$. Obviously if Θ is not either equal to a graph subgroup or trivial, then $K(\Lambda)^\Theta = \emptyset$. If Θ is a graph subgroup conjugate to Γ (Γ possibly corresponding to the trivial homomorphism), then

$$K(\Lambda)^\Theta_n = \left(\Lambda^\Theta_\Gamma\right)^{(n+1)}$$

and thus $K(\Lambda)^\Theta$ is the bar construction on Λ^Θ_Γ, therefore contractible. The last subgroup for us to check is the trivial group. That is, we must show that $K(\Lambda)$ has underlying points contractible. To show this, we prove the following lemma.

LEMMA 4.5. *Let S, T be two finite sets. Then the following simplicial set is contractible.*

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots \\
S^{\times 3} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & S^{\times 2} \times T \arrow[dl] \arrow[d, shift left=1] \arrow[d, shift right=1] & S \times T^{\times 2} \arrow[dl, shift left=1] \arrow[dl, shift right=1] \arrow[d] & T^{\times 3} \arrow[dl, shift left=2] \arrow[dl] \arrow[dl, shift right=2] \\
S^{\times 2} \arrow[d, shift left=1] \arrow[d, shift right=1] & S \times T \arrow[dl] \arrow[d] & T^{\times 2} \arrow[dl, shift left=1] \arrow[dl, shift right=1] & \\
S & T & &
\end{tikzcd}$$

(degeneracy maps not drawn for clarity)

PROOF. Let B denote the simplicial set above B^i denote the subcomplex formed by the parts with at most i copies of T. So B^0 is the first column above (i.e., the bar construction on S), B^1 is the first two columns, B^2 is the first three, and so on, with

$$B^0 \subset B^1 \subset B^2 \subset \cdots \subset \mathrm{colim}_{n\to\infty} B^n \simeq B$$

Clearly B^0 is contractible, and each B^i/B^{i-1} is contractible because

$$\left(\;\cdots \Rrightarrow S^{\times 2} \times T^{\times i} \rightrightarrows S \times T^{\times i} \;\right) \to T$$

is an equivalence of simplicial sets. It follows that B is contractible. $\square$

Apply the lemma when $S = \Lambda_0$ and $T = \Lambda_\Gamma$ to get $K(\Lambda)_\Gamma$ has contractible underlying points. Then because $K(\Lambda)$ is the union of all of the $K(\Lambda)_\Gamma$'s intersecting at a copy of $E\Lambda$ (which is also contractible), it follows that $K(\Lambda)$ has contractible underlying points, as desired. Finally, $K(\Lambda_1 \times \Lambda_2) \to K(\Lambda_1) \times K(\Lambda_2)$ is an equivalence because of the commutative diagram

$$\begin{tikzcd} K(\Lambda_1 \times \Lambda_2) \arrow[r] \arrow[d, "\simeq"] & K(\Lambda_1) \times K(\Lambda_2) \arrow[d, "\simeq"] \\ E_{C_p}(\Lambda_1 \times \Lambda_2) \arrow[r, "\simeq"] & E_{C_p}(\Lambda_1) \times E_{C_p}(\Lambda_2) \end{tikzcd}$$

□

5. p-localization of spaces and spectra

Just as there is a p-localization functor on spaces, there is similarly a p-localization functor on G-spaces (and hence, G-spectra). Since we will be primarily interested in p-localized spectra (such as $H\underline{\mathbb{F}}_p \wedge X$ for any G-spectrum X), we include the definitions here for completeness. See **[27]** II.2 and II.3 for proofs.

DEFINITION 5.1. *A nilpotent space Z is called* p-local *if it satisfies the following equivalent properties.*

- *$\pi_*(Z)$ is a p-local graded abelian group.*
- *If $X \to Y$ is an isomorphism on all p-local cohomology theories, then $[Y, Z] \to [X, Z]$ is a bijection.*
- *$H_*(Z; \mathbb{Z})$ is p-local.*

THEOREM 5.2. *Let X be a nilpotent space. Then there is a* p-localization $X \to X_{(p)}$ *that is unique up to homotopy, which satisfies the following equivalent properties.*

- *$[X_{(p)}, Z] \to [X, Z]$ is a bijection whenever Z is p-local.*
- *$X \to X_{(p)}$ is an isomorphism on all p-local cohomology theories.*

- *$\pi_* X \to \pi_* X_{(p)}$ is a (p)-localization.*
- *$H_*(X;\mathbb{Z}) \to H_*(X_{(p)};\mathbb{Z})$ is a (p)-localization.*

DEFINITION 5.3. *A Mackey functor is called* p-local *if it takes values in p-local abelian groups. A nilpotent G-space Z is called* p-local *if it satisfies the following equivalent properties.*

- *Each Z^H is a p-local space.*
- *If $X \to Y$ is an isomorphism on $\underline{M}$-cohomology for every p-local cohomological coefficient system $\underline{M}$, then $[Y,Z]_G \to [X,Z]_G$ is a bijection.*

THEOREM 5.4. *(**[27]** II.3.2) Let X be a nilpotent G-space. Then there is a* p-localization *$X \to X_{(p)}$ that is unique up to homotopy, which satisfies the following equivalent properties.*

- *$[X_{(p)}, Z] \to [X, Z]$ is a bijection for all p-local G-spaces Z.*
- *$X \to X_{(p)}$ is an isomorphism on $\underline{M}$-cohomology for every p-local cohomological coefficient system $\underline{M}$.*
- *Each $X^H \to (X_{(p)})^H$ is p-localization.*

p-localization clearly extends to spectra, and in particular, p-locality of a G-spectrum may be detected on the geometric fixed points under all subgroups H.

6. The Steinberg idempotent at p-groups

Let G be any p-group, and $\mathrm{GL}_k = \mathrm{GL}_k \mathbb{F}_p$. The goal of this section is to prove the following.

PROPOSITION 6.1. *Let X be a pointed $(G \times \mathrm{GL}_k)$-space. Then $E\mathrm{GL}_k \hookrightarrow E_G\mathrm{GL}_k$ induces an equivalence of G-spaces*

$$(\mathbf{B}_k^{\diamondsuit} \wedge X) \wedge_{\mathrm{GL}_k} (E\mathrm{GL}_k)_+ \to (\mathbf{B}_k^{\diamondsuit} \wedge X) \wedge_{\mathrm{GL}_k} (E_G\mathrm{GL}_k)_+$$

If we desuspend both sides $k-1$ times (say, in spectra), then the object on the left should be thought of as the regular old Steinberg summand of X which doesn't worry about the G-action, while the object on the right is the G-*equivariant* version of the Steinberg summand. The upshot of the proposition above is that, in the category of G-spaces (or G-spectra) with GL_k-action, these are the same. This result relies on the assumption that G is a p-group.

Thus, there is no ambiguity in writing $\epsilon_k X$ to mean either notion of the Steinberg idempotent applied to X. In particular,

Corollary 6.2.

$$\underline{\mathbb{F}}_p[\epsilon_k X] \simeq \mathrm{St}_k \otimes_{\mathbb{F}_p[\mathrm{GL}_k]} \underline{\mathbb{F}}_p[X]$$

Proof of Proposition 3.16. The map is obviously an equivalence on underlying points, so it suffices to prove it is an equivalence on fixed points. The map on fixed points is

$$(\mathbf{B}_k^{\Diamond} \wedge X^G)_{h\mathrm{GL}_k} \to \bigvee_{\Gamma} \left((\mathbf{B}_k^{\Diamond})^{\Gamma} \wedge X^{\Gamma}\right)_{hC_{\mathrm{GL}_k}(\Gamma)}$$

where the right side is a wedge sum over conjugacy classes of graph subgroups $\Gamma \subset G \times \mathrm{GL}_k$, and the map is the inclusion of the summand corresponding to the graph $G \subset G \times \mathrm{GL}_k$. This is an equivalence by the following lemma. $\square$

Lemma 6.3. *Let V be a finite dimensional vector space over a finite field of characteristic p. Let $U \subset \mathrm{GL}(V)$ be a nontrivial unipotent subgroup. Then $(\mathbf{B}_V)^U$ is $C_{\mathrm{GL}(V)}(U)$-equivariantly contractible.*

Proof. Note that $(\mathbf{B}_V)^U$ is the poset of nontrivial U-subrepresentations of V. First, we will consider the case where V is an indecomposable U-representation. Consider the subalgebra $R \subset \mathbb{F}[\mathrm{GL}(V)]$ generated by the elements $\{u-1 | u \in U\}$. Define $V^1 = RV$, and $V^i = RV^{i-1}$. We get a descending sequence

$$V = V^0 \supsetneq V^1 \supsetneq \cdots \supsetneq V^k \supsetneq 0$$

This sequence reaches 0 (and is therefore strictly descending) because R is nilpotent (because it is when U is a maximal unipotent subgroup of GL_V). Note that $V^1 \neq 0$, because U is a nontrivial group, so $k \geq 1$. Moreover, each V^i is clearly preserved by the action of $C_{\mathrm{GL}(V)}(U)$.

We claim that for any nontrivial U-subrepresentation $W \subset V$, $W \cap V^1 \neq 0$. Because U is unipotent, W has a U-fixed vector w. By definition, V^k is the set of vectors which are fixed by every element of U, so $w \in V^k$, which implies $w \in V^1$, so indeed $W \cap V^1 \neq 0$. Therefore, we have a $C_{\mathrm{GL}(V)}(U)$-equivariant map $(\mathbf{B}_V)^U \to (\mathbf{B}_V)^U$ sending $W \mapsto V^1 \cap W$, which by standard techniques about posets, is a homotopy equivalence. The image of this map is clearly $C_{\mathrm{GL}(V)}(U)$-equivariantly contractible. Thus, $(\mathbf{B}_V)^U$ is $C_{\mathrm{GL}(V)}(U)$-equivariantly contractible.

Now, we will address the general case. Let $V = V_1 \oplus \cdots \oplus V_n$ be a decomposition into indecomposable representations of U. For each V_i, we have a descending filtration

$$V_i = V_i^0 \supsetneq V_i^1 \supsetneq \cdots \supsetneq V_i^{k_i} \supsetneq 0$$

with $k_i \geq 1$. We will show that $V^1 := V_1^1 \oplus \cdots \oplus V_n^1$ intersects every element of $(\mathbf{B}_V)^U$ nontrivially, and it is preserved by $C_{\mathrm{GL}(V)}(U)$. The first of these conditions is easy. For any $W \in (\mathbf{B}_V)^U$, its projection W_i onto V_i is either trivial, or intersects V_i^1 nontrivially. Therefore, for at least one i, W_i intersects V_i^1 nontrivially, and therefore, $W \cap V^1 \neq 0$.

The second condition is a bit trickier to prove. The V_i's fall into several isomorphism classes (among all possible isomorphism classes of indecomposable U-modules). Let us call the possible isomorphism classes $1, 2, \ldots, m$. Let us suppose that there are j_1 pieces among $V_1, \ldots, V_n$ of type 1, j_2 of type 2, and so on. Then

$$C_{\mathrm{GL}(V)}(U) \simeq (\Sigma_{j_1} \times \cdots \times \Sigma_{j_m}) \ltimes \left(\prod_{i=1}^{n} C_{\mathrm{GL}(V_i)}(U) \right)$$

$$\simeq \prod_{\ell=1}^{m} \left(\Sigma_{j_\ell} \ltimes C_{\mathrm{GL}(M_\ell)}(U)^{j_\ell}\right)$$

where M_ℓ is a module of type ℓ. Clearly V^1 is preserved by each $C_{\mathrm{GL}(V_\ell)}(U)$, and thus is preserved by each $C_{\mathrm{GL}(M_\ell)}(U)^{j_\ell}$ factor. V^1 is also preserved by each Σ_{j_ℓ} factor, as these permutations just swap the V_i's that are of the same type. Thus, V^1 is $C_{\mathrm{GL}(V)}(U)$-invariant.

Therefore, just as in the indecomposable case, $(\mathbf{B}_V)^U$ is equivariantly contractible, thanks to the existence of the element V^1. $\square$

CHAPTER 4

Symmetric Powers of the Equivariant Sphere

Let $G = C_p$. It is well-known that the symmetric power filtration can be used in the equivariant setting to generate a filtration for $H\underline{\mathbb{Z}}$ - for example, see [**34**] and [**17**]. In this section, we generalize the work of chapters 2.1, 2.3, and 2.5 by constructing filtrations converging to $H\underline{\mathbb{Z}}$ and $H\underline{\mathbb{F}}_p$ whose cofibers are equivariant suspension spectra, and then explicitly computing the cofibers and the ring structure. The main goal of this chapter is to prove the following two theorems

THEOREM 0.4. *(10.1) There is a filtration (arising from symmetric powers)*

$$S^0 \simeq D_{C_p}(0) \to D_{C_p}(1) \to D_{C_p}(2) \to \cdots \to H\underline{\mathbb{F}}_p$$

where $M_{C_p}(k) := \Sigma^{-k} D_{C_p}(k)/D_{C_p}(k-1)$ is the Steinberg summand of an equivariant classifying space, namely

$$\theta_k : M_{C_p}(k) \xrightarrow{\simeq} \epsilon_k B_{C_p}(\mathbb{Z}/p)^k$$

for every k. Moreover, these isomorphisms relate the respective natural product structures.

$$\begin{tikzcd}
M_{C_p}(i) \wedge M_{C_p}(j) \arrow[r] \arrow[d] & M_{C_p}(i+j) \arrow[d] \\
\epsilon_i B_{C_p}(\mathbb{Z}/p)^i \wedge \epsilon_j B_{C_p}(\mathbb{Z}/p)^j \arrow[r] & \epsilon_{i+j} B_{C_p}(\mathbb{Z}/p)^{i+j}
\end{tikzcd}$$

THEOREM 0.5. *(11.1) There is a graded decomposition*

$$H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p \simeq \bigvee_{k \geq 0} \underline{\mathbb{F}}_p[\Sigma^k M_{C_p}(k)]$$

where $\underline{\mathbb{F}}_p[D_{C_p}(n)]$ is the first $n+1$ summands. The zero-th summand is the unit, and $H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p$ is generated as an $H\underline{\mathbb{F}}_p$-algebra by the zero-th and first summands.

With these two theorems in hand, we can explicitly compute $H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p$, which is the subject of the next chapter. This chapter is significantly more technical than the previous ones, so it helps to keep the overall outline and goals in mind. The outline of the chapter is as follows.

- (4.1) We define the filtrations for $H\underline{\mathbb{Z}}$ and $H\underline{\mathbb{F}}_p$ coming from the symmetric powers - in particular, taking underlying points recovers the ordinary symmetric power filtrations for $H\mathbb{Z}$ and $H\mathbb{F}_p$.
- (4.2-4.5) We develop a method for computing the *geometric fixed points* of these filtrations - in particular, we show that there is a splitting

$$\Phi^{C_p} D_{C_p}(k) \simeq D(k) \vee D_{\text{free}}(k)$$

  where $D_{\text{free}}(k) \simeq (D(k-1) \wedge \Sigma B\mathbb{Z}/p_+)$. This comes from a more general cofiber sequence for $\Phi^G \mathrm{Sp}^n_G$. We also analyze how the product maps $D_{C_p}(i) \wedge D_{C_p}(j) \to D_{C_p}(i+j)$ behave with respect to this splitting.
- (4.6-4.7) We prove 6.1, i.e. that an analogous splitting occurs on the geometric fixed points of $\epsilon_k B_{C_p}(\mathbb{Z}/p)^k$, namely

$$\Phi^{C_p} \epsilon_k B_{C_p}(\mathbb{Z}/p)^k \simeq \epsilon_k B(\mathbb{Z}/p)^k \vee (\epsilon_k B_{C_p}(\mathbb{Z}/p)^k)_{\text{free}}$$

  where $(\epsilon_k B_{C_p}(\mathbb{Z}/p)^k)_{\text{free}} \simeq (\epsilon_{k-1} B(\mathbb{Z}/p)^{k-1} \wedge B\mathbb{Z}/p_+)$. We then analyze how the product maps behave with respect to this splitting, and show there's a *compatibility* between the splitting for $M_{C_p}(k)$ and the splitting for $\Phi^G \epsilon_k B_{C_p}(\mathbb{Z}/p)^k$ (7.1).
- (4.8) We compute the p-localized cofibers in the symmetric power filtration for $H\underline{\mathbb{Z}}$ - in particular, the only nontrivial cofibers are at powers of p.

- (4.9-4.10) We explicitly construct an isomorphism $\theta_1 : \epsilon_k B_{C_p}(\mathbb{Z}/p) \to M_{C_p}(1)$. Then we use this map and the product structure to generate a map $\theta_k : M_{C_p}(k) \to \epsilon_k B_{C_p}(\mathbb{Z}/p)^k$ which induces an equivalence on underlying and geometric fixed points. Thus we obtain 10.1.
- (4.11) We show that this filtration splits after smashing with $H\underline{\mathbb{F}}_p$ - in particular, we prove 11.1.
- (4.12) This is an Appendix for the chapter, containing classical definitions and proofs of some auxiliary technical statements.

In order to reduce notational clutter, we have made two conventions throughout this chapter:

- Except in 4.1 and 4.2, G will always denote the cyclic group C_p of order p.
- $\Delta_k = (\mathbb{Z}/p)^k$ is an elementary abelian p-group of rank k.
- Whenever we write BH for any group H, we mean the classifying space *with a disjoint basepoint.*

1. The Symmetric Power Filtration

Let G be any finite group. When X is a pointed G-space and M is a discrete G-module, one can explicitly construct a G-space $M \otimes X$ whose equivariant $RO(G)$-graded homotopy groups are naturally isomorphic to the Bredon homology of X with coefficients in M ([**13**]). If we let $M = \mathbb{Z}$ with the trivial action, the resulting space has a filtration by the symmetric powers of X. Considering $X = S^V$ as V ranges over all representation spheres, we get a filtration of G-spectra

$$\mathbb{S}_G \to \mathrm{Sp}^2\mathbb{S}_G \to \mathrm{Sp}^3\mathbb{S}_G \to \cdots \to \mathrm{Sp}^\infty\mathbb{S}_G \simeq H\underline{\mathbb{Z}}$$

Here, $\mathrm{Sp}^n(\mathbb{S}_G)$ is the spectrum whose V-th space is $\mathrm{Sp}^n(S^V)$ (the spectrum maps arise in the obvious way). For brevity, we will denote $\mathrm{Sp}^n\mathbb{S}_G$ by Sp^n_G, and call this the *equivariant symmetric power filtration.* In particular, note that taking underlying points gives $\Phi^e \mathrm{Sp}^n_G = \mathrm{Sp}^n$. This filtration has been studied by [**34**] and [**17**].

The homomorphism $p : \mathbb{Z} \to \mathbb{Z}$ defines a cofiber sequence $H\underline{\mathbb{Z}} \to H\underline{\mathbb{Z}} \to H\underline{\mathbb{F}}_p$. The induced map on symmetric powers is the *p-replication map* $d : \mathrm{Sp}^n_G \to \mathrm{Sp}^{pn}_G$. This can also be thought of as arising on a space level from the product map

$$\mathrm{Sp}^n X \wedge \mathrm{Sp}^p S^0 \to \mathrm{Sp}^{pn} X$$

by using p copies of the nontrivial point in S^0. Thus, we obtain a filtration

$$\mathbb{S}_G \to \mathrm{Sp}^2_{p,G} \to \mathrm{Sp}^3_{p,G} \to \mathrm{Sp}^4_{p,G} \to \cdots H\underline{\mathbb{F}}_p$$

whose n-th stage is obtained from Sp^n_G by quotienting out the part generated by the images of all $d : \mathrm{Sp}^i_G \to \mathrm{Sp}^{pi}_G$ for $i \leq n/p$. [1] We call this the *mod p equivariant symmetric power filtration.*

Analogous to notation from the nonequivariant case, we define

$$L_G(k) := \Sigma^{-k} \mathrm{Sp}^{p^k}_G / \mathrm{Sp}^{p^{k-1}}_G$$

$$D_G(k) := \mathrm{Sp}^{p^k}_{p,G} \qquad M_G(k) := \Sigma^{-k} \mathrm{Sp}^{p^k}_{p,G} / \mathrm{Sp}^{p^{k-1}}_{p,G}$$

The next few sections will be focused on computing the *geometric fixed points* of $\mathrm{Sp}^n_{C_p}$ in terms of the ordinary symmetric power filtration.

2. The Fixed Points of $\mathrm{Sp}^n X$

For this section, let G denote an arbitrary finite group. Let X be a pointed G-space, and let T be a finite G-set of size n. Then let $X_T \subset (\mathrm{Sp}^n X)^G \subset (\mathrm{Sp}^\infty X)^G$ be the subspace of unordered tuples $(x_1, \ldots, x_n)$ which, as a G-set, are isomorphic

[1] When $\alpha \geq 0$ is not an integer, we define

$$\mathrm{Sp}^\alpha(X) := \mathrm{Sp}^{\lfloor \alpha \rfloor}(X)$$

to T. Clearly $(\mathrm{Sp}^n X)^G = \bigcup_{|T| \leq n} X_T$, and these X_T's intersect trivially, but there are attaching data.

Define a partial ordering $\succ$ on isomorphism classes of finite G-sets as follows. If K, H are subgroups of G such that some conjugate of K is a subgroup of H (which we will denote by $K \leq H$), then define

$$G/H \succ [H : K] \cdot (G/K)$$

and extend this relation by declaring that if $T \succ U$, then $T \sqcup S \succ U \sqcup S$. Then the closure of X_T is contained in $\bigcup_{S \preceq T} X_S$, which itself is a pointed subspace of $(\mathrm{Sp}^\infty X)^G$. This is a consequence of the fact that the H-isotropy part of X, $X(H)$, is contained in $X^H \simeq \bigcup_{K \leq H} X(K)$.

Write down a filtration of $(\mathrm{Sp}^\infty X)^G$, where each successive stage attaches another X_T. This is a filtration indexed by the finite G-sets T, in any order subject to the condition that if $S \preceq T$, then X_S is attached at an earlier stage than X_T. Then the cofiber at the stage which attaches X_T is

$$\mathrm{Pr}_G^T X \simeq \left(\bigcup_{S \preceq T} X_S \right) / \left(\bigcup_{S \prec T} X_S \right)$$

which we call the T-*primitives* of X. An equivalent definition is

$$\mathrm{Pr}_G^T X = \overline{X_T} / (\overline{X_T} - X_T)$$

where the closure is taken in $(\mathrm{Sp}^\infty X)^G$. $\mathrm{Pr}_G^T(-)$ is a functor from G-spaces to spaces - it may be thought of as a subquotient of the functor $(\mathrm{Sp}^\infty(-))^G$. Some of its key properties are proven below.

PROPOSITION 2.1. *Call a finite G-set* isotypic *if its orbits are all isomorphic. If $T = T_1 \sqcup \ldots \sqcup T_r$ where $r \geq 1$ and each T_i is isotypic and nontrivial, then*

$$\mathrm{Pr}_G^T X \simeq \mathrm{Pr}_G^{T_1} X \wedge \cdots \wedge \mathrm{Pr}_G^{T_r} X$$

If $T = m \cdot S$ where S is a single orbit, then

$$\mathrm{Pr}_G^T X \simeq \mathrm{Pr}_1^m(\mathrm{Pr}_G^S X)$$

PROOF. For the first property, notice that $X_T \simeq X_{T_1} \times \cdots \times X_{T_r}$. The result now follows by standard point-set topology reasons.

For the second property, it is clear that for any pointed space Y, $\mathrm{Pr}_1^m(Y) \simeq Y^{\wedge m}/\Sigma_m$. Thus, letting $A = \bigcup_{R \preceq S} X_R$ and $B = \bigcup_{R \prec S} X_R$,

$$\mathrm{Pr}_1^m(\mathrm{Pr}_G^S X) \simeq (A/B)^{\wedge m}/\Sigma_m \simeq \mathrm{Sp}^m A/(B \wedge \mathrm{Sp}^{m-1} A)$$

The top of this quotient is formed from all X_U where U is a sum of m G-sets below or equal to S - this is the same as G-sets below or equal to T. The bottom is formed from all U among these with at least one component strictly below S - this is the same as G-sets strictly below T. The result follows. $\square$

$(-)_T$ (resp. $\mathrm{Pr}_G^T(-)$) is a functor from G-spaces (resp. pointed G-spaces) to spaces (resp. pointed spaces). Let V be any real representation of G. Then we have maps

$$V^G \times X_T \to (S^V \wedge X)_T$$

$$S^{V^G} \wedge \mathrm{Pr}_G^T X \to \mathrm{Pr}_G^T(S^V \wedge X)$$

Thus, Pr_G^T extends to a functor from genuine G-spectra to spectra. So just as the $\{\mathrm{Pr}_G^T(-)\}_T$ are the cofibers in a filtration for $(\mathrm{Sp}^\infty(-))^G$ on spaces, they are the cofibers in a filtration for $\Phi^G \mathrm{Sp}^\infty(-)$ on spectra. In the next section, we will study the behavior of $\mathrm{Pr}_G^T(\mathbb{S}_G)$, and use this to compute $\Phi^G \mathrm{Sp}_G^n$.

3. Primitives in the stable setting

For the rest of this chapter, let $G = C_p$. In this section, we completely compute $\mathrm{Pr}_G^T(\mathbb{S}_G)$ for T an arbitrary G-set. Most notably, when T is nonisotypic, the resulting piece is contractible. The final result is as follows.

PROPOSITION 3.1. *If T is non-isotypic, $\mathrm{Pr}_G^T(\mathbb{S}_G) \simeq *$. For the isotypic G-sets,*

$$\mathrm{Pr}_G^{n(G/G)}(\mathbb{S}_G) \simeq \Sigma^\infty(\mathbf{P}_n^\Diamond \wedge S^n)_{h\Sigma_n} \simeq \mathrm{Sp}^n/\mathrm{Sp}^{n-1}$$

$$\mathrm{Pr}_G^{n(G/1)}(\mathbb{S}_G) \simeq \Sigma^\infty(\Sigma B\mathbb{Z}/p \wedge (\mathbf{P}_n^\Diamond \wedge S^n)_{h\Sigma_n}) \simeq (\mathrm{Sp}^n/\mathrm{Sp}^{n-1}) \wedge \Sigma B\mathbb{Z}/p$$

We first address the case where T consists of a single orbit. (After this, we will apply 2.1 to get to the general case.)

PROPOSITION 3.2. [2]

$$\mathrm{Pr}_G^{G/G}(S^{\ell\rho_G}) \simeq S^\ell \qquad \textit{and} \qquad \mathrm{Pr}_G^{G/1}(S^{\ell\rho_G}) \overset{p\ell-1}{\simeq} S^\ell \wedge \Sigma B\mathbb{Z}/p$$

Thus,[3]

$$\mathrm{Pr}_G^{G/G}(\mathbb{S}_G) := \Phi^G\mathbb{S}_G \simeq \mathbb{S} \qquad \textit{and} \qquad \mathrm{Pr}_G^{G/1}(\mathbb{S}_G) \simeq \mathbb{S} \wedge \Sigma B\mathbb{Z}/p$$

PROOF. $S^{\ell\rho_G} \simeq S^\ell \wedge S^{\ell\overline{\rho}_G}$ with S^ℓ being the fixed points. Thus, the first result follows immediately. Towards the second, $\mathrm{Pr}_G^{G/1}(S^{\ell\rho_G}) \simeq S^\ell \wedge \mathrm{Pr}_G^{G/1}(S^{\ell\overline{\rho}_G})$. $S^{\ell\overline{\rho}_G}$ has fixed points S^0 (corresponding to the antipodal points $0, \infty$), and

$$S^{\ell\overline{\rho}_G}/S^0 \simeq S^1 \wedge U_+$$

[2]Here, we say $X \overset{m}{\simeq} Y$ if there is a map $X \to Y$ which induces an equivalence on $\pi_0, \pi_1, \ldots, \pi_m$. Such a map is called an *m-equivalence*. Some basic properties about this are proven in the Appendix for this chapter - see 12.2. In particular, if $X = \{X_m\}$ and $Y = \{Y_m\}$ are spectra, and we have $f : X \to Y$ such that $f_m : X_m \to Y_m$ is a $(m + g(m))$-equivalence for some function $g : \mathbb{N} \to \mathbb{Z}$ which grows without bound, then f is an equivalence. We say the maps $\{f_m\}$ give a *stable equivalence* and write $X_m \overset{s}{\simeq} Y_m$.

[3]Remember, for this chapter we use $B\mathbb{Z}/p$ to denote the classifying space with a *disjoint basepoint!*

where U is the (unpointed) unit sphere in $S^{\ell\overline{\rho}_G}$. U is a sphere of dimension $(p-1)\ell-1$ with free G-action, and thus

$$S^1 \wedge (U/G)_+ \overset{(p-1)\ell-1}{\simeq} S^1 \wedge B\mathbb{Z}/p$$

is a stable equivalence. The second result follows. □

Now consider the case where T is an arbitrary isotypic G-set, i.e. $T = n(G/G)$ or $n(G/1)$ for some n. We use the following proposition, which follows from the same argument as Proposition 7.2 of [**2**]. For completeness, the argument is reproduced in the Appendix. Let $\mathbf{P}_n$ be the partition poset of $\{1, \ldots, n\}$.

PROPOSITION 3.3. *Let Z be any pointed space. Then we have a stable equivalence* $\mathrm{Pr}_1^n(S^\ell \wedge Z) \overset{s}{\simeq} S^\ell \wedge Z \wedge (\mathbf{P}_n^\diamondsuit \wedge S^n)_{h\Sigma_n}$.

PROOF. See 12.1. □

From this, we can perform the computation when T is isotypic.

COROLLARY 3.4.

$$\mathrm{Pr}_G^{n(G/G)}(S^{\ell\rho_G}) \simeq \mathrm{Pr}_1^n(\mathrm{Pr}_G^{(G/G)}(S^{\ell\rho_G})) \overset{s}{\simeq} S^\ell \wedge (\mathbf{P}_n^\diamondsuit \wedge S^n)_{h\Sigma_n}$$

$$\mathrm{Pr}_G^{n(G/1)}(S^{\ell\rho_G}) \simeq \mathrm{Pr}_1^n(\mathrm{Pr}_G^{(G/1)}(S^{\ell\rho_G})) \overset{s}{\simeq} S^\ell \wedge \Sigma B\mathbb{Z}/p \wedge (\mathbf{P}_n^\diamondsuit \wedge S^n)_{h\Sigma_n}$$

PROOF. The first result follows immediately, so we just have to prove the second, i.e. compute $\mathrm{Pr}_G^{n(G/1)}(S^{\ell\rho_G})$. As we know, $\mathrm{Pr}_G^{G/1}(S^{\ell\rho_G})$ is $(2\ell-1)$-equivalent to $S^\ell \wedge \Sigma B\mathbb{Z}/p$. Therefore, we have a $(2\ell-1)$-connected map (which is therefore a $(2\ell-2)$-equivalence)

$$\mathrm{Pr}_G^{n(G/1)}(S^{\ell\rho_G}) \simeq \mathrm{Pr}_1^n(\mathrm{Pr}_G^{G/1}(S^{\ell\rho_G})) \overset{2\ell-2}{\simeq} S^\ell \wedge \Sigma B\mathbb{Z}/p \wedge (\mathbf{P}_n^\diamondsuit \wedge S^n)_{\tilde{h}\Sigma_n}$$

It follows that

$$\mathrm{Pr}_G^{n(G/1)}(\mathbb{S}_G) \simeq \Sigma^\infty(\Sigma B\mathbb{Z}/p) \wedge (\mathbf{P}_n^\diamondsuit \wedge S^n)_{\tilde{h}\Sigma_n}$$

$\square$

All that is left is an analysis of $\mathrm{Pr}_G^T(\mathbb{S}_G)$ when T is non-isotypic. Consider

$$\mathrm{Pr}_G^{m(G/G)\sqcup n(G/1)}(S^{\ell\rho_G}) \simeq \mathrm{Pr}_G^{m(G/G)}(S^{\ell\rho_G}) \wedge \mathrm{Pr}_G^{n(G/1)}(S^{\ell\rho_G})$$

by 2.1. We have shown that each of the pieces on the right is $(2\ell-1)$-equivalent to an ℓ-fold suspension. Therefore, the right hand side is $(2\ell-2)$-equivalent to a 2ℓ-suspension. So the right hand side is $(2\ell-2)$-connected, and so when we pass to the stable setting, $\mathrm{Pr}_G^{m(G/G)\sqcup n(G/1)}(\mathbb{S}_G) \simeq *$.

4. Decomposition of $\Phi^G(\mathrm{Sp}^n\mathbb{S}_G)$

We now use the results on stable primitives to compute $\Phi^G\mathrm{Sp}^n_G$. The goal of this section is to prove the following theorem.

THEOREM 4.1. *There is a cofiber sequence*

$$\mathrm{Sp}^n \longrightarrow \Phi^G\mathrm{Sp}^n_G \longrightarrow \mathrm{Sp}^{n/p} \wedge \Sigma B\mathbb{Z}/p$$

where the attaching map $\mathrm{Sp}^{n/p} \wedge B\mathbb{Z}/p \to \mathrm{Sp}^n$ is the product of the counit map $B\mathbb{Z}/p \to S^0$ and the p-replication map $\mathrm{Sp}^{n/p} \to \mathrm{Sp}^n$. As n varies, these cofiber sequences can be connected by using the maps in the nonequivariant and equivariant symmetric power filtrations.

$$\begin{array}{ccccc}
\mathrm{Sp}^{n-1} & \longrightarrow & \Phi^G\mathrm{Sp}^{n-1}_G & \longrightarrow & \mathrm{Sp}^{(n-1)/p} \wedge \Sigma B\mathbb{Z}/p \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Sp}^n & \longrightarrow & \Phi^G\mathrm{Sp}^n_G & \longrightarrow & \mathrm{Sp}^{n/p} \wedge \Sigma B\mathbb{Z}/p
\end{array}$$

PROOF. Let $X = S^{\ell\rho_G}$. Define $Y_0 = (\mathrm{Sp}^n X)^G$ and for each i, consider the successive cofiber sequences

$$Z_i \hookrightarrow Y_i \to Y_{i+1}$$

where

- Y_i is the part of $(\mathrm{Sp}^n X)^G$ formed by quotienting away X_T for every T containing *fewer than* i free orbits.
- $Z_i \subset Y_i$ is the subspace formed by the X_T's with *exactly* i free orbits (in particular, $Z_0 = \mathrm{Sp}^n(X^G)$).

We can assemble these data into a diagram

$$\begin{tikzcd} Z_0 \arrow[d] & Z_1 \arrow[d] & Z_2 \arrow[d] & \cdots & Z_{k-1} \arrow[d] & \\ Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & \cdots & Y_{k-1} \arrow[r] & Z_k \end{tikzcd} \tag{4.1}$$

where $k = \lfloor n/p \rfloor$.

(1) **Constructing the cofiber sequence.** We will show, by reverse induction on i, that

$$Y_i \overset{s}{\simeq} \Sigma B\mathbb{Z}/p \wedge (\mathrm{Sp}^k/\mathrm{Sp}^{i-1})(S^\ell)$$

for $i \geq 1$, and thus, $Y_1 \overset{s}{\simeq} \Sigma B\mathbb{Z}/p \wedge \mathrm{Sp}^k(S^\ell)$. This will imply that the desired cofiber sequence comes from $Z_0 \to Y_0 \to Y_1$ as $\ell \to \infty$.

The inclusion

$$\Sigma B\mathbb{Z}/p \wedge (\mathrm{Sp}^i/\mathrm{Sp}^{i-1})(S^\ell) \overset{s}{\simeq} \mathrm{Pr}_G^{i(G/1)}(X) \hookrightarrow Z_i$$

is a stable equivalence, because by 3.1, the primitives for any *nonisotypic* G-set are stably contractible. Let us consider how these Z_i's attach together. Consider a point in $X_{m(G/1)}$ consisting of m free G-orbits, $S_1, \dots, S_m \subset X$. This can equivalently be viewed as m points $x_1, \dots, x_m$ in $S^\ell \wedge (S^1 \wedge (U/G)_+)$. We approach the boundary of $X_{m(G/1)}$ when the points in some S_i converge - this corresponds to x_i moving to the basepoint. The resulting boundary point $X_{(m-1)(G/1)\sqcup p(G/G)}$ retracts onto the point $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m)$ in

$\mathrm{Pr}_1^{m-1}(S^\ell \wedge (S^1 \wedge (U/G)_+))$. Thus, it is clear that the pieces $Z_i \simeq \Sigma B\mathbb{Z}/p \wedge (\mathrm{Sp}^i/\mathrm{Sp}^{i-1})(S^\ell)$ attach together to give $Y_i \overset{s}{\simeq} \Sigma B\mathbb{Z}/p \wedge (\mathrm{Sp}^k/\mathrm{Sp}^{i-1})(S^\ell)$, as desired.

(2) **Showing the cofiber sequence behaves well with the inclusions $\mathrm{Sp}_G^{n-1} \to \mathrm{Sp}_G^n$.** If we let Y_i', Z_i' be the corresponding spaces for $(\mathrm{Sp}^{n-1}X)^G$, then there are clearly maps $Y_i' \to Y_i$ and $Z_i' \to Z_i'$ which make the corresponding copies of 4.1 commute. $Z_0' \to Z_0$ is the obvious map $\mathrm{Sp}^{n-1}(X^G) \to \mathrm{Sp}^n(X^G)$. Meanwhile, the inclusion $Z_i' \to Z_i$ has cofiber $\mathrm{Pr}_G^{i(G/1)}(X) \wedge \mathrm{Pr}_G^{n-ip}(X)$, which is stably contractible, and therefore this map is an equivalence unless $n = ip$ (in which case $Z_i' \simeq *$). It is now clear that the map $Y_1' \to Y_1$ is stably equivalent to the obvious map $\Sigma B\mathbb{Z}/p \wedge \mathrm{Sp}^{\lfloor (n-1)/p \rfloor}(S^\ell) \to \Sigma B\mathbb{Z}/p \wedge \mathrm{Sp}^{\lfloor n/p \rfloor}(S^\ell)$.

(3) **Computing the attaching map $\mathrm{Sp}^{\lfloor n/p \rfloor}(B\mathbb{Z}/p \wedge S^\ell) \to \mathrm{Sp}^n(S^\ell)$.** This will follow from an analysis of the attach map of the cofiber sequence for $n = p$.

$$\mathrm{Sp}^p(S^\ell) \to (\mathrm{Sp}^p(S^{\ell\rho_G}))^G \to S^\ell \wedge (S^1 \wedge U_+)/G$$

As we move towards the nontrivial boundary of the $S^\ell \wedge (S^1 \wedge U_+)/G$ part, the $(S^1 \wedge U_+)/G$ coordinate goes to the basepoint. When this happens, the point we get in $\mathrm{Sp}^p(S^\ell)$ is a sum of p copies of the S^ℓ coordinate. Therefore, the attaching map is the composite

$$\begin{tikzcd}
S^\ell \wedge (U/G)_+ \arrow[r, dashed] \arrow[d] & \mathrm{Sp}^p(S^\ell) \\
S^\ell \wedge S^0 \arrow[r, "\simeq"] & S^\ell \arrow[u, "d"']
\end{tikzcd}$$

□

5. Decomposition of $\Phi^G(\mathrm{Sp}_p^n\mathbb{S}_G)$ and Product structure

As in **[31]**, let $D(k) := \mathrm{Sp}_p^{p^k}$ and $M(k) := \Sigma^{-k}\mathrm{Sp}_p^{p^k}/\mathrm{Sp}_p^{p^{k-1}}$. Similarly, let us define $D_G(k) = \mathrm{Sp}_{p,G}^{p^k}$ and $M_G(k) = \Sigma^{-k}\mathrm{Sp}_{p,G}^{p^k}/\mathrm{Sp}_{p,G}^{p^{k-1}}$. Then an argument completely analogous to that given for the ordinary equivariant symmetric power filtration, gives us a cofiber sequence

$$D(k) \to \Phi^G D_G(k) \to D_{\mathrm{free}}(k)$$

where $D_{\mathrm{free}}(k) \simeq D(k-1) \wedge \Sigma B\mathbb{Z}/p$ is the part of $\Phi^G D_G(k)$ formed from the free G-sets. This cofiber sequence is also functorial in the inclusions $D_G(k-1) \to D_G(k)$. The goal of this section is to first prove that these cofiber sequences are *split*, and then to analyze how the product maps $D_G(i) \wedge D_G(j) \to D_G(i+j)$ behave on geometric fixed points. Proving they split is quite easy:

PROPOSITION 5.1. *This cofiber sequence splits, i.e.* $\Phi^G D_G(k) \simeq D(k) \vee D_{\mathrm{free}}(k)$ *and the inclusions* $D_G(k-1) \to D_G(k)$ *respect this splitting. (Hence, there is also a splitting* $\Phi^G M_G(k) \simeq M(k) \vee M_{\mathrm{free}}(k)$ *with* $M_{\mathrm{free}}(k) \simeq M(k-1) \wedge B\mathbb{Z}/p$*.)*

PROOF. For any G-space X and any G-set T, let $X_{T,p}$ denote the image of X_T under the map $\mathrm{Sp}^\infty(X)^G \to \mathrm{Sp}_p^\infty(X)^G$. That is, $X_{T,p}$ consists of tuples of points $(x_1, \dots, x_n) \in \mathrm{Sp}_p^\infty(X)^G$ such that the action of G acts on these points as the G-set T.

In $\mathrm{Sp}_p^\infty(X)^G$, $X_{T,p}$ has boundary contained in $\bigcup_{S \subset T} X_{S,p}$. This is because when the points corresponding to a free orbit in T come together at a point in X^G, the result is p times that point, which is zero in $\mathrm{Sp}_p^\infty(X)$. Since $D_{\mathrm{free}}(k)$ is formed entirely out of the $(S^{\ell\rho_G})_{T,p}$ for T a *free* G-set, its boundary is the basepoint in $\mathrm{Sp}_p^\infty(S^{\ell\rho_g})$. $\square$

Note: Another way to see that this cofiber sequence splits, is to consider the diagram

$$\begin{tikzcd}
\mathrm{Sp}^{p^{k-1}} \wedge B\mathbb{Z}/p \arrow[r] \arrow[d] & \mathrm{Sp}^{p^k} \arrow[r] \arrow[d] & \Phi^G \mathrm{Sp}^{p^k}_G \arrow[r] \arrow[d] & \mathrm{Sp}^{p^{k-1}} \wedge \Sigma B\mathbb{Z}/p \arrow[d] \\
\Sigma^{-1} D_{\mathrm{free}}(k) \arrow[r] & D(k) \arrow[r] & \Phi^G D_G(k) \arrow[r] & D_{\mathrm{free}}(k)
\end{tikzcd}$$

The attaching map $D(k-1) \wedge B\mathbb{Z}/p \to D(k)$ is zero, because the p-replication map $d : D(k-1) \to D(k)$ is zero.

Now, we study the *product* structure $D_G(i) \wedge D_G(j) \to D_G(i+j)$ and the effect this structure has on the splitting of the geometric fixed points.

LEMMA 5.2. *Let $\mu_{1,1}$ denote the product map $D_{\mathrm{free}}(1) \wedge D_{\mathrm{free}}(1) \to D_{\mathrm{free}}(2)$. Then, after we identify $D_{\mathrm{free}}(1) \simeq \Sigma^\infty \Sigma B\mathbb{Z}/p$ and $D_{\mathrm{free}}(2) \simeq D(1) \wedge \Sigma B\mathbb{Z}/p$, there is a commutative diagram*

$$\begin{tikzcd}
\Sigma^\infty(\Sigma B\mathbb{Z}/p \wedge \Sigma B\mathbb{Z}/p) \arrow[r, "\mu_{1,1}"] \arrow[d, "\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]"'] & D(1) \wedge \Sigma B\mathbb{Z}/p \arrow[d] \\
\Sigma^\infty(\Sigma B\mathbb{Z}/p \wedge \Sigma B\mathbb{Z}/p) \arrow[r] & \Sigma^\infty(\Sigma B\Sigma_p \wedge \Sigma B\mathbb{Z}/p)
\end{tikzcd}$$

with the bottom horizontal map arising from the group inclusion $\mathbb{Z}/p \hookrightarrow \Sigma_p$, and the right vertical map coming from the projection $D(1) \to M(1) \simeq \Sigma^\infty \Sigma B\Sigma_p$.

PROOF. If we have a p-tuple $\overline{x} = \{x, gx, \ldots, g^{p-1}x\} \subset S^{\ell\rho}$ and a p-tuple $\overline{y} = \{y, gy, \ldots, g^{p-1}y\}$ in $S^{m\rho}$, we can take their product to get a p^2-tuple $\overline{x} \wedge \overline{y} = \{(g^i x, g^j y)\}_{0 \le i,j \le p-1}$ in $S^{(\ell+m)\rho}$. Considering the diagonal action of G on $S^{(\ell+m)\rho}$, this can be thought of as the union of the G-orbits generated by $\{(g^i x, y)\}_{0 \le i \le p-1}$. If either $x \in (S^{\ell\rho})^G \simeq S^\ell$ or $y \in (S^{m\rho})^G \simeq S^m$, then this p^2-tuple consists of p copies of the same orbit - and if either lies outside these fixed points, then the diagonal action of G is free. Therefore, we have a map

$$(S^{\ell\rho}/S^{\ell})/G \wedge (S^{m\rho}/S^{m})/G \to \mathrm{Sp}_p^p((S^{(\ell+m)\rho}/S^{\ell+m})/G)$$

which lands in the free part. Here, we have used the observation that an orbit $\overline{x} = \{x, gx, \ldots, g^{p-1}x\}$ in $S^{\ell\rho}/S^{\ell}$ is the same as a single point in $(S^{\ell\rho}/S^{\ell})/G$, and likewise for m and $\ell + m$. As we let $\ell, m \to \infty$, this multiplication can be summed up in the following diagram

$$\begin{tikzcd}
\Sigma^\infty(\Sigma B\mathbb{Z}/p \wedge \Sigma B\mathbb{Z}/p) \arrow[d, "\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]"'] & \\
\Sigma^\infty(\Sigma B\mathbb{Z}/p \wedge \Sigma B\mathbb{Z}/p) \arrow[r, "\eta"] & \mathrm{Sp}_p^p(\Sigma^\infty \Sigma B\mathbb{Z}/p)
\end{tikzcd}$$

The coordinate change of the first map ensures that the second copy of $\Sigma B\mathbb{Z}/p$ is the diagonal copy of our original wedge sum: this gets identified with the copy of $\Sigma^\infty\Sigma B\mathbb{Z}/p$ coming from $(S^{(\ell+m)\rho}/S^{\ell+m})/G$. Note that $\mathrm{Sp}_p^p(\Sigma^\infty\Sigma B\mathbb{Z}/p) \simeq D(1) \wedge \Sigma B\mathbb{Z}/p$. If we now compose with the quotient map $D(1) \to D(1)/D(0) \simeq \Sigma B\Sigma_p$, we get the natural map (dotted)

$$\begin{tikzcd}
\Sigma^\infty(\Sigma B\mathbb{Z}/p \wedge \Sigma B\mathbb{Z}/p) \arrow[r, "\eta"] \arrow[dr, dashed] & D(1) \wedge \Sigma B\mathbb{Z}/p \arrow[d] \\
& \Sigma^\infty(\Sigma B\Sigma_p \wedge \Sigma B\mathbb{Z}/p)
\end{tikzcd}$$

coming from the inclusion of groups $\mathbb{Z}/p \hookrightarrow \Sigma_p$ (on classifying spaces, this map is independent of which inclusion we pick). This is true because this composite takes our p^2-tuple of points in $S^{\ell\rho}/S^{\ell} \wedge S^{m\rho}/S^{m}$, mods out by the diagonal action of G (which corresponds to the second copy of $B\mathbb{Z}/p$), and then forgets the order of the resulting p-tuple, thus modding out by the action of Σ_p on such tuples.

$\square$

PROPOSITION 5.3. *After identifying* $D_{\mathrm{free}}(k) \simeq D(k-1) \wedge D_{\mathrm{free}}(1)$ *for all* k, *the product map* $\mu_{i,j} : D_{\mathrm{free}}(i) \wedge D_{\mathrm{free}}(j) \to D_{\mathrm{free}}(i+j)$ *is the composite (dotted)*

$$\begin{tikzcd}
(D(i-1) \wedge D_{\mathrm{free}}(1)) \wedge (D(j-1) \wedge D_{\mathrm{free}}(1)) \arrow[d, "\mu_{1,1}"] \arrow[dr, dashed, "\mu_{i,j}"] & \\
D(i-1) \wedge D(j-1) \wedge D(1) \wedge D_{\mathrm{free}}(1) \arrow[r] & D(i+j-1) \wedge D_{\mathrm{free}}(1)
\end{tikzcd}$$

PROOF. This is straightforward. □

6. Splitting of orbit types for Steinberg Summands

Remember that we write $\Delta_k = (\mathbb{Z}/p)^k$ to mean an elementary abelian p group of rank k. The objective of this section is to prove the following splitting, analogous to splitting for $\Phi^G D_G(k)$ in the last section. (We will use 12.5, which is a lemma about restriction of the Bruhat-Tits building to a parabolic subgroup.)

PROPOSITION 6.1.

$$((\mathbf{B}_k^{\lozenge})_{h_G \mathrm{Aff}_k})^G \simeq (\mathbf{B}_k^{\lozenge})_{h\mathrm{Aff}_k} \vee ((\mathbf{B}_{k-1}^{\lozenge})_{h\mathrm{Aff}_{k-1}} \wedge \Sigma B\mathbb{Z}/p)$$

Or, written another way,

$$\Phi^G(\epsilon_k B_G \Delta_k) \simeq \epsilon_k B\Delta_k \vee (\epsilon_k B\Delta_k)_{\mathrm{free}}$$

where $(\epsilon_k B\Delta_k)_{\mathrm{free}} \simeq (\epsilon_{k-1} B\Delta_{k-1} \wedge B\mathbb{Z}/p)$. *The first component corresponds to the trivial summand of* $(B_G\Delta_k)^G \simeq \bigvee_{\Delta_k} B\Delta_k$, *and the second component corresponds to all of the nontrivial summands.*

PROPOSITION 6.2.

$$((\mathbf{B}_k^{\lozenge} \wedge S^{\rho_k})_{h_G \mathrm{Aff}_k})^G \simeq (\mathbf{B}_k^{\lozenge} \wedge S^{\rho_k})_{h\mathrm{Aff}_k} \vee ((\mathbf{B}_{k-1}^{\lozenge} \wedge S^{\rho_{k-1}})_{h\mathrm{Aff}_{k-1}} \wedge \Sigma B\mathbb{Z}/p)$$

where ρ_k is the real regular representation of Δ_k, and $\rho_{k-1} \subset \rho_k$ is the subrepresentation fixed by the last basis vector e_k of Δ_k. Written another way,

$$\Phi^G(\epsilon_k B_G \Delta_k^{\overline{\rho}_k}) \simeq \epsilon_k B\Delta_k^{\overline{\rho}_k} \vee (\epsilon_{k-1} B\Delta_{k-1}^{\overline{\rho}_{k-1}} \wedge B\mathbb{Z}/p)$$

Proof. *(of 6.1)* There is a formula for the fixed points on the left side

$$((\mathbf{B}_k^{\lozenge})_{hG\mathrm{Aff}_k})^G \simeq \bigvee_{[f:G\to \mathrm{Aff}_k]} ((\mathbf{B}_k^{\lozenge})^{\mathrm{im} f})_{hC_{\mathrm{Aff}_k}(\mathrm{im} f)}$$

Recall that $\mathrm{Aff}_k \simeq \Delta_k \rtimes \mathrm{GL}_k$, where $\Delta_k \simeq (\mathbb{Z}/p)^k$ is the group of translations. For any such $f : G \to \mathrm{Aff}_k$, call the projection of $\mathrm{im} f$ onto GL_k by U. By (Chapter 3, 6.3), if U is nontrivial then $(\mathbf{B}_k^{\lozenge})^U$ is $C_{\mathrm{GL}_k}(U)$-equivariantly contractible, and thus $(\mathbf{B}_k^{\lozenge})^{\mathrm{im} f}$ is $C_{\mathrm{Aff}_k}(\mathrm{im} f)$-equivariantly contractible, because $C_{\mathrm{Aff}_k}(\mathrm{im} f) \subset \Delta_k \rtimes C_{\mathrm{GL}_k}(U)$. It follows that the only nontrivial wedge summands are those corresponding to maps $f : G \to \Delta_k$.

Such maps come in two conjugacy classes - the trivial map, and the nontrivial maps. For the trivial map, the resulting summand is $(\mathbf{B}_k^{\lozenge})_{h\mathrm{Aff}_k}$ - this gives us the first summand on the right hand side of 6.1. Let us analyze the other summand. First any nontrivial map $f : G \to \Delta_k$, and pick a basis $e_1, \dots, e_k$ for Δ_k such that the image of f is spanned by e_k. Then

$$C_{\mathrm{Aff}_k}(\mathrm{im} f) \simeq \Delta_k \rtimes (U_{k-1} \rtimes \mathrm{GL}_{k-1})$$

where GL_{k-1} is the group of linear transformations on $\Delta_{k-1} = \langle e_1, \dots, e_{k-1} \rangle$ and U_{k-1} is the group of unipotent matrices associated to the flag $0 \subset \langle e_k \rangle \subset \Delta_k$. That is, U_{k-1} consists of the lower triangular unpotent matrices concentrated in the bottom row. Then

$$(\mathbf{B}_k^{\lozenge})_{hC_{\mathrm{Aff}_k}(\mathrm{im} f)} \simeq (\mathbf{B}_k^{\lozenge} \wedge B\Delta_k)_{h(U_{k-1} \rtimes \mathrm{GL}_{k-1})}$$

Let P_{k-1} denote the parabolic associated to the flag $0 \subset \langle e_k \rangle \subset \Delta_k$. Then as a P_{k-1}-space,

$$\mathbf{B}_k^{\Diamond} \wedge B\Delta_k \simeq \left(\bigvee_{X \perp \langle e_k \rangle} (\mathbf{B}_X^{\Diamond} * \mathbf{B}_{\langle e_k \rangle}^{\Diamond}) \right) \wedge B\Delta_k \simeq \bigvee_{X \perp \langle e_k \rangle} (\mathbf{B}_X^{\Diamond} \wedge BX) * (\mathbf{B}_{\langle e_k \rangle}^{\Diamond}) \wedge B\langle e_k \rangle)$$

by 12.5. Here, the wedge sum is taken over all X transverse to $\langle e_k \rangle$, and P_{k-1} permutes the summands. Therefore, when we take homotopy orbits under $U_{k-1} \rtimes \mathrm{GL}_{k-1}$, we get

$$\simeq (\mathbf{B}_{k-1}^{\Diamond} \wedge B\Delta_{k-1})_{h\mathrm{GL}_{k-1}} * (\mathbf{B}_{\langle e_k \rangle}^{\Diamond} \wedge B\langle e_k \rangle) \simeq (\mathbf{B}_{k-1}^{\Diamond} \wedge B\Delta_{k-1})_{h\mathrm{GL}_{k-1}} \wedge \Sigma B\mathbb{Z}/p$$

which gives us the second summand in 6.1. $\square$

PROOF. *(of 6.2.)* The proof is analogous to the previous one. The fixed points on the left side decompose into a wedge sum with two summands. The summand corresponding to the trivial map $f : G \to \Delta_k$ is $(\mathbf{B}_k^{\Diamond} \wedge S^{\rho_k})_{h\mathrm{Aff}_k}$, and the summand corresponding to the nontrivial map $f : G \to \Delta_k$ with image spanned by e_k is

$$(\mathbf{B}_k^{\Diamond} \wedge (S^{\rho_k})^{\mathrm{im} f})_{hC_{\mathrm{Aff}_k}(\mathrm{im} f)} \simeq (\mathbf{B}_k^{\Diamond} \wedge (S^{\rho_{k-1}})_{h\Delta_k} \rangle)_{h(U_{k-1} \rtimes \mathrm{GL}_{k-1})}$$

As a P_{k-1}-space,

$$\mathbf{B}_k^{\Diamond} \wedge (S^{\rho_{k-1}})_{h\Delta_k} \simeq \left(\bigvee_{X \perp \langle e_k \rangle} (\mathbf{B}_X^{\Diamond} * \mathbf{B}_{\langle e_k \rangle}^{\Diamond}) \right) \wedge (S^{\rho_{k-1}})_{h\Delta_k}$$

$$\simeq \bigvee_{X \perp \langle e_k \rangle} (\mathbf{B}_X^{\Diamond} \wedge (S^{\rho_{k-1}})_{hX}) * (\mathbf{B}_{\langle e_k \rangle}^{\Diamond} \wedge B\langle e_k \rangle)$$

Therefore, when we take homotopy orbits under $U_{k-1} \rtimes \mathrm{GL}_{k-1}$, we get

$$\simeq (\mathbf{B}_{k-1}^{\Diamond} \wedge (S^{\rho_{k-1}})_{h\Delta_{k-1}})_{h\mathrm{GL}_{k-1}} * (\mathbf{B}_{\langle e_k \rangle}^{\Diamond} \wedge B\langle e_k \rangle)$$

which is equivalent to the second summand. $\square$

7. Product structure on geometric fixed points

We showed in 5.1 that there is a splitting

$$\Phi^G M_G(k) \simeq M(k) \vee M_{\text{free}}(k)$$

with $M_{\text{free}}(k) \simeq M(k-1) \wedge B\mathbb{Z}/p$. We also showed in 6.1 that there is a splitting

$$\Phi^G \epsilon_k B_G\Delta_k \simeq \epsilon_k B\Delta_k \vee (\epsilon_k B\Delta_k)_{\text{free}}$$

with $(\epsilon_k B\Delta_k)_{\text{free}} \simeq \bigvee_{v\in\Delta_k^\times} B\Delta_k \simeq \epsilon_{k-1}B\Delta_k \wedge B\mathbb{Z}/p$. The goal of this section is to relate these two splittings by proving the following theorem.

THEOREM 7.1. *Let $\alpha_k : \epsilon_k B\Delta_k \to M(k)$ be the isomorphism of ([31], Theorem A) and let $\beta_k : (\epsilon_k B\Delta_k)_{\text{free}} \to M_{\text{free}}(k)$ be the isomorphism $\beta_k = \alpha_{k-1} \wedge B\mathbb{Z}/p$. Then there are commutative diagrams*

$$\begin{CD}
\epsilon_i B\Delta_i \wedge \epsilon_j B\Delta_j @>>> \epsilon_{i+j}B\Delta_{i+j}\\
@V{\alpha_i\wedge\alpha_j}VV @VV{\alpha_{i+j}}V\\
M(i)\wedge M(j) @>>> M(i+j)
\end{CD}
\qquad
\begin{CD}
\epsilon_i B\Delta_i \wedge (\epsilon_j B\Delta_j)_{\text{free}} @>>> (\epsilon_{i+j}B\Delta_{i+j})_{\text{free}}\\
@V{\alpha_i\wedge\beta_j}VV @VV{\beta_{i+j}}V\\
M(i)\wedge M_{\text{free}}(j) @>>> M_{\text{free}}(i+j)
\end{CD}$$

$$\begin{CD}
(\epsilon_i B\Delta_i)_{\text{free}} \wedge (\epsilon_j B\Delta_j)_{\text{free}} @>>> (\epsilon_{i+j}B\Delta_{i+j})_{\text{free}}\\
@V{\beta_i\wedge\beta_j}VV @VV{\beta_{i+j}}V\\
M_{\text{free}}(i)\wedge M_{\text{free}}(j) @>>> M_{\text{free}}(i+j)
\end{CD}$$

where the horizontal maps arise from taking the geometric fixed points of the product maps $\epsilon_i B_G\Delta_i \wedge \epsilon_j B_G\Delta_j \to \epsilon_{i+j}B_G\Delta_{i+j}$ and $M_G(i)\wedge M_G(j) \to M_G(i+j)$.

PROOF. We have already analyzed the product structure on the geometric fixed points of $D_G(k)$, so it suffices to compute the product structure on the geometric fixed points $\Phi^G\epsilon_k B_G\Delta_k$. The product maps $B_G\Delta_i \wedge B_G\Delta_j \to B_G\Delta_{i+j}$ induce, on fixed points

$$\bigvee_{v_i \in \Delta_i} B\Delta_i \wedge \bigvee_{v_j \in \Delta_j} B\Delta_j \to \bigvee_{v_{i+j} \in \Delta_{i+j}} B\Delta_{i+j}$$

where elements v_i, v_j are multiplied in the obvious way. Thus, the product map $\Phi^G(\epsilon_i B_G \Delta_i) \wedge \Phi^G(\epsilon_j B_G \Delta_j) \to \Phi^G(\epsilon_{i+j} B_G \Delta_{i+j})$ induces the four maps

$$\epsilon_i B\Delta_i \wedge \epsilon_j B\Delta_j \to \epsilon_{i+j} B\Delta_{i+j}$$

$$\begin{tikzcd}
\epsilon_i B\Delta_i \wedge (\epsilon_j B\Delta_j)_{\text{free}} \arrow[dr] & \\
(\epsilon_i B\Delta_i)_{\text{free}} \wedge \epsilon_j B\Delta_j \arrow[r] & (\epsilon_{i+j} B\Delta_{i+j})_{\text{free}} \\
(\epsilon_i B\Delta_i)_{\text{free}} \wedge (\epsilon_j B\Delta_j)_{\text{free}} \arrow[ur] &
\end{tikzcd}$$

(1) The first map is just the obvious map arising from the Steinberg summands of nonequivariant classifying spaces - call this map $m_{i,j}$.

(2) To describe the second and third map, we note that in the equivalence

$$\epsilon_k \bigvee_{v \in \Delta_k} B\Delta_k \simeq \epsilon_k B\Delta_k \vee \epsilon_k \bigvee_{v \in \Delta_k^\times} (B\Delta_{k-1} \wedge B\langle v \rangle)$$

$$\simeq \epsilon_k B\Delta_k \vee (\epsilon_{k-1} B\Delta_{k-1} \wedge B\mathbb{Z}/p)$$

the generator of $B\mathbb{Z}/p$ comes from the element $v \in \Delta_k^\times$ (above, $\Delta_{k-1} = \Delta_k/\langle v \rangle$). Now the second and third maps are easy to describe: for example, thanks to our observation above about the origin of $B\mathbb{Z}/p$, we have a commutative diagram

$$\begin{tikzcd}
\epsilon_i B\Delta_i \wedge \epsilon_j \bigvee_{v_j \neq 0} B\Delta_j \arrow[r] \arrow[d, "\simeq"] & \epsilon_{i+j} \bigvee_{v_{i+j} \neq 0} B\Delta_i \wedge B\Delta_j \arrow[d, "\simeq"] \\
\epsilon_i B\Delta_i \wedge \epsilon_{j-1} B\Delta_{j-1} \wedge B\mathbb{Z}/p \arrow[r] & \epsilon_{i+j-1}(B\Delta_i \wedge B\Delta_{j-1}) \wedge B\mathbb{Z}/p
\end{tikzcd}$$

with the map along the bottom coming from the product.

(3) On the level of summands, if $v_i, v_j \neq 0$ then we have maps[4]

$$(B\Delta_{i-1} \wedge B\langle v_i\rangle) \wedge (B\Delta_{j-1} \wedge B\langle v_j\rangle) \to B\Delta_{i+j-1} \wedge B\langle v_i v_j\rangle$$

where $\Delta_{i-1} = \Delta_i/\langle v_i\rangle$, and $\Delta_{j-1}, \Delta_{i+j-1}$ are defined similarly. The fourth map $(\epsilon_i B\Delta_i)_{\text{free}} \wedge (\epsilon_j B\Delta_j)_{\text{free}} \to (\epsilon_{i+j} B\Delta_{i+j})_{\text{free}}$ is therefore equivalent to the composite

$$\begin{array}{c}
(\epsilon_{i-1}B\Delta_{i-1} \wedge B\langle v_i\rangle) \wedge (\epsilon_{j-1}B\Delta_{j-1} \wedge B\langle v_j\rangle) \\
\downarrow \\
\epsilon_{i-1}B\Delta_{i-1} \wedge \epsilon_{j-1}B\Delta_{j-1} \wedge B\Delta_1 \wedge B\langle v_i v_j\rangle \\
\downarrow \\
(\epsilon_{i-1}B\Delta_{i-1} \wedge \epsilon_{j-1}B\Delta_{j-1} \wedge \epsilon_1 B\Delta_1) \wedge B\langle v_i v_j\rangle \\
\downarrow \\
\epsilon_{i+j-1}B\Delta_{i+j-1} \wedge B\langle v_{i+j}\rangle
\end{array}$$

where the first map comes from $B\langle v_i\rangle \wedge B\langle v_j\rangle \xrightarrow{\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]} B\langle v_i\rangle \wedge B\langle v_i v_j\rangle$, the second from the projection $B\Delta_1 \to \epsilon_1 B\Delta_1$, and the third from the product.

Combining this analysis with 5.2, we get the theorem.

□

8. Cofibers for $H\mathbb{Z}$

Analogous to the nonequivariant case, there is an equivalence ([**17**])

$$\mathrm{Sp}^n_G/\mathrm{Sp}^{n-1}_G \simeq \Sigma^\infty(\mathbf{P}^\lozenge_n \wedge S^n) \wedge_{\Sigma_n} (E_G\Sigma_n)_+$$

In this section, we prove the following theorem, analogous to [**3**].

[4] $v_i v_j \in \Delta_{i+j}$ denotes the direct sum of $v_i \in \Delta_i$ and $v_j \in \Delta_j$.

THEOREM 8.1. *If n is not a power of p, then there is a p-local equivalence $(\mathbf{P}_n^\Diamond \wedge S^n) \wedge_{\Sigma_n} (E_G\Sigma_n)_+ \simeq *$. If $n = p^k$, then there is a p-local equivalence*

$$\Sigma^\infty(\mathbf{B}_k^\Diamond \wedge S^{p^k}) \wedge_{\mathrm{Aff}_k} (E_G\mathrm{Aff}_k)_+ \to \Sigma^\infty(\mathbf{P}_n^\Diamond \wedge S^n) \wedge_{\Sigma_n} (E_G\Sigma_n)_+$$

This theorem is interesting in its own right, but we will need to use it in a minor way to prove 9.1

PROOF. First, suppose n is not a power of p. We will show that $(\mathbf{P}_n^\Diamond \wedge S^n) \wedge_{\Sigma_n} (E_G\Sigma_n)_+$ has both underlying and fixed points p-locally contractible. On underlying points, this reduces to the nonequivariant case, so we will address the fixed points.

$$((\mathbf{P}_n^\Diamond \wedge S^n)_{h_G\Sigma_n})^G \simeq \bigvee_{[f:G\to\Sigma_n]} ((\mathbf{P}_n^\Diamond)^{\mathrm{im} f} \wedge (S^n)^{\mathrm{im} f})_{hC_{\Sigma_n}(\mathrm{im} f)}$$

$$\simeq \bigvee_{|T|=n} (\mathbf{P}_T^\Diamond \wedge S^{|T/G|})_{h\Sigma_T}$$

where the wedge sum is taken over isomorphism classes of G-sets T, and $\mathbf{P}_T$ and Σ_T are the poset of equivariant partitions and the group of equivariant automorphisms, respectively, of T. By Lemma 7.1 of [**1**], $\mathbf{P}_T$ is Σ_t-equivariantly contractible unless T is isotypic, so we may restrict the wedge sum to isotypic T. There are then two cases to consider: T trivial, and T free. For T a trivial G set, the corresponding summand is just $(\mathbf{P}_n^\Diamond \wedge S^n)_{h\Sigma_n}$, which is contractible unless n is a power of p.

For T free - by Proposition 9.1 of [**1**], if $T = (n/p)S$, where S is the free orbit, then there is a homotopy equivalence

$$(\Sigma_S)^{n/p} \wedge_{\Sigma_S} \Sigma\mathbf{P}_S * \mathbf{P}_{n/p}^\Diamond \to \mathbf{P}_T^\Diamond$$

The partition poset of a free G-orbit has just two elements (the discrete and indiscrete partitions), so this implies that $(\mathbf{P}_T^\Diamond)_{hG^{n/p}} \simeq \Sigma B\mathbb{Z}/p \wedge \mathbf{P}_{n/p}$. The group of automorphisms of a free G-orbit is G, so $\Sigma_T \simeq G^{n/p} \rtimes \Sigma_{n/p}$. Thus, if T is a free G-set with

n/p orbits, we have an equivalence

$$(\mathbf{P}_T^{\Diamond} \wedge S^{n/p})_{h\Sigma_T} \simeq \left((\mathbf{P}_T^{\Diamond})_{hG^{n/p}} \wedge S^{n/p}\right)_{h\Sigma_{n/p}} \simeq \left((\Sigma B\mathbb{Z}/p \wedge \mathbf{P}_{n/p}^{\Diamond}) \wedge S^{n/p}\right)_{h\Sigma_{n/p}}$$

Elements of $\Sigma_{n/p}$ permute the copies of S, and therefore act trivially on $\Sigma B\mathbb{Z}/p$. Thus, this summand is p-locally trivial unless n/p (and therefore n) is a power of p. This proves that if n is not a power of p, $(\mathbf{P}_n^{\Diamond} \wedge S^n)_{h_G\Sigma_n}$ is p-locally contractible.

Now let $n = p^k$. The map

$$(\mathbf{B}_k^{\Diamond} \wedge S^{p^k}) \wedge_{\mathrm{Aff}_k} (E_G\mathrm{Aff}_k)_+ \to (\mathbf{P}_n^{\Diamond} \wedge S^n) \wedge_{\Sigma_n} (E_G\Sigma_n)_+$$

arises from the inclusion $B_{p^k} \to \mathbf{P}_{p^k}$. We will prove the described map is a p-local equivalence by showing it is a p-local equivalence on both underlying and fixed points. On underlying points, the result follows from **[2]**, so we only need to check fixed points. On the left side, we have a decomposition (6.1)

$$((\mathbf{B}_k^{\Diamond} \wedge S^{\rho_k})_{h_G\mathrm{Aff}_k})^G \simeq \bigvee_{[f:G\to\mathrm{Aff}_k]} ((\mathbf{B}_k^{\Diamond})^{\mathrm{im}f} \wedge (S^{\rho_k})^{\mathrm{im}f})_{C_{\mathrm{Aff}_k}(\mathrm{im}f)}$$

$$\simeq (\mathbf{B}_k^{\Diamond} \wedge S^{\rho_k})_{h\mathrm{Aff}_k} \vee ((\mathbf{B}_{k-1}^{\Diamond} \wedge S^{\rho_{k-1}})_{h\mathrm{Aff}_{k-1}} \wedge \Sigma B\mathbb{Z}/p)$$

with the first summand corresponding to the trivial class $f : G \to \Delta_k$, and the second summand corresponding to the nontrivial class. On the right side, we have a splitting

$$((\mathbf{P}_n^{\Diamond} \wedge S^n)_{h_G\Sigma_n})^G \simeq \bigvee_{|T|=n} ((\mathbf{P}_T^{\Diamond}) \wedge S^{|T/G|})_{h\Sigma_T}$$

$$\simeq (\mathbf{P}_n^{\Diamond} \wedge S^n)_{h\Sigma_n} \vee ((\mathbf{P}_{n/p}^{\Diamond} \wedge S^{n/p})_{h\Sigma_{n/p}} \wedge \Sigma B\mathbb{Z}/p)$$

with the first summand corresponding to the trivial G-set and the second corresponding to the free G-set. It is therefore clear that the map 8 respects these decompositions, and induces a p-local equivalence. $\square$

9. $H\underline{\mathbb{F}_p}$ - the first cofiber

Recall that $M_G(k) = D_G(1)/D_G(0) = \Sigma^{-k}\mathrm{Sp}^{p^k}_{p,G}/\mathrm{Sp}^{p^{k-1}}_{p,G}$. The goal of this section is to explicitly prove the following theorem:

THEOREM 9.1. $M_G(1) \simeq \Sigma^\infty B_G\mathrm{Aff}_1 \simeq \epsilon_1\Sigma^\infty B_G\Delta_1$

We do this by directly analyzing this spectrum evaluated at the representation sphere associated to $\ell\rho_G$, and then observing the stable behavior as $\ell \to \infty$. We need the following two simple lemmas about the symmetric group Σ_p. Their proofs are given in the Appendix.

LEMMA 9.2. *12.7 Let $\sigma \in \Sigma_p$ be a p-cycle, and let $H \subset \Sigma_p$ be a nontransitive subgroup which is normalized by σ. Then H is the trivial group. Therefore, if $f : G \to \Sigma_p$ is a nontrivial homomorphism, then any subgroup of $G \times \Sigma_p$ which properly contains Γ_f has intersection with Σ_p transitive.*

LEMMA 9.3. *12.8 Let β denote the reduced standard representation of Σ_p (of dimension $p-1$), and let $\Gamma \subset G \times \Sigma_p$ be a subgroup. Then $(\rho_G \otimes \beta)^\Gamma = 0$ if and only if $\Gamma \cap \Sigma_p$ acts transitively on $\{1, 2, \ldots, p\}$.*

We now prove the theorem in two steps.

PROPOSITION 9.4. $(\mathrm{Sp}^p/\mathrm{Sp}^1)(S^{\ell\rho_G})/d(S^{\ell\rho_G})$ *is stably p-locally equivalent to $S^{\ell\rho_G} \wedge S^1 \wedge (E_G\mathcal{F})/\Sigma_p$, where $\mathcal{F}$ is the collection of nontransitive subgroups of Σ_p, and $E_G\mathcal{F}$ is the $(G \times \Sigma_p)$-space associated to subgroups $\Gamma \subset G \times \Sigma_p$ such that $\Gamma \cap \Sigma_p \in \mathcal{F}$.*

PROOF. We know that $(\mathrm{Sp}^n/\mathrm{Sp}^{n-1})(S^{\ell\rho_G}) \simeq S^{n\ell\rho_G}/\Sigma_n$ is stably p-locally trivial unless n is a power of p (by 8.1), and therefore

$$\mathrm{Sp}^p_p(S^{\ell\rho_G})/S^{\ell\rho_G} \simeq \left(S^{p\ell\rho_G}/d(S^{\ell\rho_G})\right)/\Sigma_p \simeq S^{\ell\rho_G} \wedge (S^{\ell\rho_G\beta}/S^0)/\Sigma_p$$

where β is the reduced standard representation of Σ_p (of dimension $p-1$). As a $(G \times \Sigma_p)$-space,

$$S^{\ell\rho_G\beta}/S^0 \simeq S^1 \wedge U(\ell\rho_G\beta)$$

where $U(-)$ denotes the unit sphere in a real representation. Let $\Gamma \subseteq G \times \Sigma_p$. By 12.8, $U(\ell\rho_G\beta)^\Gamma$ is empty if $\Gamma \cap \Sigma_p$ is transitive, and is otherwise the unit sphere of a real vector space of dimension at least ℓ. So, stably,

$$\mathrm{colim}_{\ell\to\infty} U(\ell\rho_G\beta) \simeq E_G\mathcal{F}$$

where $\mathcal{F}$ is the collection of nontransitive subgroups of Σ_p. It follows that, stably, $\mathrm{Sp}_p^p(S^{\ell\rho_G})/d(S^{\ell\rho_G})$ is equivalent to $S^{\ell\rho_G} \wedge S^1 \wedge (E_G\mathcal{F}/\Sigma_p)$. □

PROPOSITION 9.5. *p-locally, $E_G\mathcal{F}/\Sigma_p \simeq B_G\Sigma_p \simeq B_G\mathrm{Aff}_1$.*

PROOF. There is a canonical approximation map $E_G\Sigma_p \to E_G\mathcal{F}$ (for an introduction to approximations of equivariant spaces, see [**2**], chapter 2). This gives a map $B_G\Sigma_p \to E_G\mathcal{F}/\Sigma_p$. On underlying points, this is the map $B\Sigma_p \to (E\mathcal{F}/\Sigma_p)$, which is a p-local equivalence, by [**31**].[5] On G-fixed points, we note that for any $(G \times \Sigma_p)$-space X,

$$(X/\Sigma_p)^G \simeq \left(\bigcup_{f:G\to\Sigma_p} X_f \right) /\Sigma_p$$

where $X_f \subset X$ is the subspace of X which is fixed by Γ_f, the graph of f. If $f, f' :\to \Sigma_p$ are two distinct maps, then $X_f \cap X_{f'}$ is the subspace of X fixed by the group Γ generated by both Γ_f and $\Gamma_{f'}$. Clearly $\Gamma \cap \Sigma_p$ is nontrivial, so if $X = E_G\Sigma_p$, the intersections $X_f \cap X_{f'}$ are trivial. But we can say more - by 12.7, $\Gamma \cap \Sigma_p$ is *transitive*, so if $X = E_G\mathcal{F}$, the intersections $X_f \cap X_{f'}$ are trivial as well. In particular, by 12.7, if $X = E_G\mathcal{F}$, no point in X_f can have isotropy group larger than Γ_f, and thus, Σ_p acts freely on this (disjoint) union.

[5] The same argument we used above, but in the nonequivariant case, shows that $\mathrm{Sp}_p^p \simeq S^1 \wedge \Sigma^\infty(E\mathcal{F}/\Sigma_p)$, and Mitchell-Priddy prove that, p-locally, $\mathrm{Sp}_p^p \simeq S^1 \wedge \Sigma^\infty B\Sigma_p \simeq S^1 \wedge \Sigma^\infty B\mathrm{Aff}_1$.

So now, it suffices to show that the map $E_G\Sigma_p \to E_G\mathcal{F}$ induces an equivalence on Γ_f-fixed points for every $f : G \to \Sigma_p$. This is clear, as

$$(E_G\Sigma_p)^{\Gamma_f} \simeq (E_G\mathcal{F})^{\Gamma_f} \simeq *$$

Finally, we need to show that $B_G\mathrm{Aff}_1 \to B_G\Sigma_p$ is a p-local equivalence. On underlying points, $B\mathrm{Aff}_1 \to B\Sigma_p$ is an equivalence by **[31]**. On fixed points, the map is

$$\bigvee_{f:G\to \mathrm{Aff}_1} BC_{\mathrm{Aff}_1}(\mathrm{im} f) \to \bigvee_{[f:G\to\Sigma_p]} BC_{\Sigma_p}(\mathrm{im} f)$$

induced by the inclusion of groups $\mathrm{Aff}_1 \hookrightarrow \Sigma_p$. There are two wedge summands on each side - corresponding to the trivial homomorphism, and the collection of nontrivial homomorphisms (which are all conjugate). There are $p-1$ nontrivial homomorphisms $G \to \mathrm{Aff}_1$, and each is centralized by the subgroup $\Delta_1 \subset \mathrm{Aff}_1$, so the left side is equivalent to $B\mathrm{Aff}_1 \vee B\Delta_1$. Meanwhile, there are $(p-1)!$ nontrivial homomorphisms $G \to \Sigma_p$, and each is centralized by its own image, so the right side is equivalence to $B\Sigma_p \vee B\Delta_1$. It is easy to see that the map is the identity $B\Delta_1 \to B\Delta_1$ on the nontrivial summands, and on the trivial summands, is the p-local equivalence $B\mathrm{Aff}_1 \to B\Sigma_p$. □

10. $\mathbf{H}\underline{\mathbb{F}_p}$ - the k-th cofiber

Recall that $\alpha_k : \epsilon_k B\Delta_k \to M(k)$ is the equivalence of (**[31]**, Theorem A), and $\beta_k : \epsilon_k \bigvee_{v\in\Delta_k^\times} B\Delta_k \to M_{\mathrm{free}}(k)$ is the equivalence $\beta_k = \alpha_{k-1} \wedge B\mathbb{Z}/p$. In this section, we will prove the following theorem

Theorem 10.1. *There is a map $\theta_k : \epsilon_k B_G\Delta_k \to M_G(k)$ which induces an equivalence on underlying points, and on geometric fixed points, induces the equivalence $\alpha_k \vee \beta_k$. Therefore, it induces an equivalence of G-spectra.*

The map is quite easy to construct. Consider the commutative diagram

$$\begin{tikzcd} \epsilon_k B_G \Delta_k \arrow[r, "\iota"] & (\epsilon_1 B_G \Delta_1)^{\wedge k} \arrow[r, "\nu"] \arrow[d, "\theta_1^{\wedge k}"] & \epsilon_k B_G \Delta_k \\ & (M_G(1))^{\wedge k} \arrow[r, "\mu"] & M_G(k) \end{tikzcd}$$

Here, ι is the inclusion of the Steinberg summand $\epsilon_k B_G \Delta_k \to B_G \Delta_k$ followed by the projection, μ and ν are the multiplication maps, and θ_1 is the isomorphism constructed in the last section. Then let $\theta_k = \mu \circ \theta \circ \iota$. Clearly $\nu \circ \iota$ is the identity map. We wish to show that θ_k is an isomorphism - we do so by showing that μ and ν induce the same map on underlying points and on geometric fixed points. On underlying points, this is due to [**31**]. So we just have to address the geometric fixed points.

We can assume that $\theta_2, \ldots, \theta_{k-1}$ satisfy the desired properties, and proceed by induction. Then we need to show that in the following diagram

$$\begin{tikzcd} \epsilon_{k-1} B_G \Delta_{k-1} \wedge \epsilon_1 B_G \Delta_1 \arrow[r] \arrow[d, "\theta_{k-1} \wedge \theta_1"] & \epsilon_k B_G \Delta_k \arrow[d, "\theta_k"] \\ M_G(k-1) \wedge M_G(1) \arrow[r] & M_G(k) \end{tikzcd}$$

the two horizontal multiplication maps induce the same map on geometric fixed points. This follows immediately from 7.1.

11. The Filtration splits

In this section, we prove that in the mod p symmetric power filtration

$$S^0 \simeq D_{C_p}(0) \to D_{C_p}(1) \to D_{C_p}(2) \to \cdots \to H\underline{\mathbb{F}}_p$$

every cofiber sequence $D_{C_p}(k-1) \to D_{C_p}(k) \to \Sigma^k M_{C_p}(k)$ splits after we smash with $H\underline{\mathbb{F}}_p$. In particular, we show the following equivariant analogue to 5.1.

THEOREM 11.1. *There is a graded decomposition*

$$H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p \simeq \bigvee_{k\geq 0} \underline{\mathbb{F}}_p[\Sigma^k M_{C_p}(k)]$$

where $\underline{\mathbb{F}}_p[D_{C_p}(n)]$ *is the first* $n+1$ *summands. The zero-th summand is the unit, and* $H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p$ *is generated as an* $H\underline{\mathbb{F}}_p$*-algebra by the zero-th and first summands.*

The proof is essentially identical to that of (Chapter 2, 5.1). We first generate a splitting of the first cofiber sequence:

$$\begin{tikzcd} \underline{\mathbb{F}}_p[S^0] \arrow[r, shift right] & \underline{\mathbb{F}}_p[D_{C_p}(1)] \arrow[l, dashed, shift right] \end{tikzcd}$$

Then we use the product structure to split the rest.[6]

PROPOSITION 11.2. *There is a splitting* $\begin{tikzcd} S^0 \wedge H\underline{\mathbb{F}}_p \arrow[r, shift right] & D_{C_p}(1) \wedge H\underline{\mathbb{F}}_p \arrow[l, dashed, shift right] \end{tikzcd}$ *.*

PROOF. The composition $S^0 \to D_{C_p}(1) \to H\underline{\mathbb{F}}_p$ is the unit map $1 : S^0 \to H\underline{\mathbb{F}}_p$. Therefore, after smashing with $H\underline{\mathbb{F}}_p$, and composing with the product map $\mu : H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p \to H\underline{\mathbb{F}}_p$, we have the commutative diagram

$$\begin{tikzcd} S^0 \wedge H\underline{\mathbb{F}}_p \arrow[r] \arrow[drr, "\mathrm{Id}"'] & D_{C_p}(1) \wedge H\underline{\mathbb{F}}_p \arrow[r] \arrow[dr, dashed] & H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p \arrow[d, "\mu"] \\ & & H\underline{\mathbb{F}}_p \end{tikzcd}$$

The dotted map provides the splitting. $\square$

Using the above map, we have a splitting $\begin{tikzcd} D_{C_p}(1) \wedge H\underline{\mathbb{F}}_p \arrow[r, shift right] & \Sigma M_{C_p}(1) \wedge H\underline{\mathbb{F}}_p \arrow[l, dashed, shift right] \end{tikzcd}$ - we denote this dotted map by t_1. Now define the map $t_k : \underline{\mathbb{F}}_p[\Sigma^k M_{C_p}(k)] \to \underline{\mathbb{F}}_p[D_{C_p}(k)]$

[6]The structure of this argument was communicated to me by my advisor, Mike Hopkins. My initial attempts at proving this splitting were quite clumsy and computational, and I was shocked at how simple his argument was.

by the composite

$$\begin{tikzcd}
\underline{\mathbb{F}}_p[\Sigma^k M_{C_p}(k)] \simeq \underline{\mathbb{F}}_p[\Sigma^k \epsilon_k B_{C_p} \Delta_k] \arrow[r, "\iota"] \arrow[d, dashed, "t_k"'] & \underline{\mathbb{F}}_p[(\Sigma \epsilon_1 B_{C_p} \Delta_1)^{\wedge k}] \arrow[d, "t_1^{\otimes k}"] \\
\underline{\mathbb{F}}_p[D_{C_p}(k)] & \underline{\mathbb{F}}_p[D_{C_p}(1)^{\wedge k}] \arrow[l, "\mu"']
\end{tikzcd}$$

Here, the top horizontal map ι is the inclusion of the Steinberg summand, and the bottom horizontal map μ comes from the product structure on the symmetric powers.

Proposition 11.3. *t_k is a monomorphism, and therefore splits the sequence*

$$\begin{tikzcd}
\underline{\mathbb{F}}_p[D_{C_p}(k-1)] \arrow[r] & \underline{\mathbb{F}}_p[D_{C_p}(k)] \arrow[r] & \underline{\mathbb{F}}_p[\Sigma^k M_{C_p}(k)] \arrow[l, dashed, bend right]
\end{tikzcd}$$

Proof.

$$\begin{tikzcd}
\underline{\mathbb{F}}_p[D_{C_p}(1)^{\wedge k}] \arrow[r, "\overline{\mu}"] \arrow[d, shift left] & \underline{\mathbb{F}}_p[D_{C_p}(k)] \arrow[d, shift right] \\
\underline{\mathbb{F}}_p[(\Sigma M_{C_p}(1))^{\wedge k}] \arrow[r, "\mu"] \arrow[u, dashed, bend left, "t_1^{\otimes k}"] & \underline{\mathbb{F}}_p[\Sigma^k M_{C_p}(k)] \arrow[u, dashed, bend right, "t_k"'] \arrow[l, dashed, bend left, "\iota"]
\end{tikzcd}$$

The non-dotted arrows clearly form a commutative diagram. We want to show that $t_k = \overline{\mu} \circ t_1^{\otimes k} \circ \iota$ is a monomorphism. It suffices to show that t_k followed by applying the downwards map on the right side of the square, is the identity. This is equivalent to proving that $t_1^{\otimes k} \circ \iota$, followed by the downwards map on the left side of the square and then μ, is the identity. This is obvious because the left side of the square is inclusion of a summand (by the previous corollary) and the bottom of the square is inclusion of the Steinberg summand. $\square$

Corollary 11.4. *The product maps*

$$\underline{\mathbb{F}}_p[\Sigma M_{C_p}(1)]^{\otimes k} \xrightarrow{\mu} \underline{\mathbb{F}}_p[\Sigma^k M_{C_p}(k)]$$

are projection onto the Steinberg summand, and hence surjective. Hence, we have a surjective map of $H\underline{\mathbb{F}}_p$-algebras

$$\bigvee_{k\geq 0} \underline{\mathbb{F}}_p[\Sigma M_{C_p}(1)]^{\otimes k} \to \bigvee_{k\geq 0} \underline{\mathbb{F}}_p[\Sigma^k M_{C_p}(k)] \simeq H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p$$

12. Appendix

Here, we include definitions and lemmas which are necessary for completeness, but are routine and would detract from the flow of the chapter.

12.0.1. *4.3: Primitives in the nonequivariant setting.* Let $\mathbf{P}$ be the poset of Young subgroups of Σ_n (i.e., of the form $\Sigma_{i_1} \times \cdots \times \Sigma_{i_m}$ where $\sum i_j = n$). The following is nearly identical to Proposition 7.2 in [**2**].

PROPOSITION 12.1. *Let Z be any pointed space. Then we have a $(2\ell-1)$-equivalence*

$$\mathrm{Pr}_1^n(S^\ell \wedge Z) := (\mathrm{Sp}^n/\mathrm{Sp}^{n-1})(S^\ell \wedge Z) \overset{2\ell-1}{\simeq} S^\ell \wedge Z \wedge (\mathbf{P}_n^\diamond \wedge S^n)_{h\Sigma_n}$$

PROOF. Let $\tilde{\mathbf{P}} = \mathbf{P} \cup \{1\}$, and $\mathcal{C} = \tilde{\mathbf{P}} \cup \{\Sigma_n\}$. First, we use the reduction

$$(\mathrm{Sp}^n/\mathrm{Sp}^{n-1})(S^\ell \wedge Z) \simeq (S^\ell \wedge Z)^{\wedge n}/\Sigma_n \simeq (S^{\ell n} \wedge Z^{\wedge n})/\Sigma_n$$

We now let $X = S^{\ell n}$ and consider the map of squares

$$\begin{tikzcd}
((X \wedge Z^{\wedge n})^\Sigma)_{\tilde{\mathbf{P}}} \arrow[r] \arrow[d] & ((X \wedge Z^{\wedge n})^\Sigma)_{\mathcal{C}} \arrow[d] & \to & ((X \wedge Z^{\wedge n})^\Sigma)_{\tilde{\mathbf{P}}} \arrow[r] \arrow[d] & ((X \wedge Z^{\wedge n})^\Sigma)_{\mathcal{C}} \arrow[d] \\
(X \wedge Z^{\wedge n})_{\tilde{\mathbf{P}}} \arrow[r] & (X \wedge Z^{\wedge n})_{\mathcal{C}} & & (*)_{\tilde{\mathbf{P}}} \arrow[r] & Y
\end{tikzcd}$$

where the top two terms are the same. The bottom left map is induced by $X \wedge Z^{\wedge n} \to *$ and Y is defined so that the right-hand square is a homotopy pushout square. The map $(X \wedge Z^{\wedge n})_{\mathcal{C}} \to Y$ is well-defined because (7.6) the left-hand square is a homotopy pushout square. If $K \subset \Sigma_n$, then $(X \wedge Z^{\wedge n})^K \simeq S^{o(K)\ell} \wedge Z^{\wedge o(K)}$, where $o(K)$ is the number of orbits of the action of K on $\mathbf{n}$. So if $K \in \tilde{\mathbf{P}}$, $(X \wedge Z^{\wedge n})^K \to (*)^K = *$ is a $(2\ell-1)$-equivalence. Therefore, the bottom left map is a $(\Sigma_n, 2\ell-1)$-equivalence. Thus, so is the bottom right map. Since $(X \wedge Z^{\wedge n})$ has $\mathcal{C}$-isotropy, $(X \wedge Z^{\wedge n})_{\mathcal{C}} \to X \wedge Z^{\wedge n}$ is a Σ_n-equivalence. The fixed point set $(X \wedge Z^{\wedge n})^{\Sigma_n}$ is $S^\ell \wedge Z$, so the right-hand square is, up to Σ-equivalence, the square

$$\begin{tikzcd}
(S^\ell \wedge Z) \times E\tilde{\mathbf{P}} \arrow[r] \arrow[d] & S^\ell \wedge Z \arrow[d] \\
E\tilde{\mathbf{P}} \arrow[r] & Y
\end{tikzcd}$$

By naturality, the upper arrow and the left-hand arrow in this square are the obvious projections. Thus, Y is Σ-equivalence to $(S^\ell \wedge Z)\#E\tilde{\mathbf{P}} \simeq S^\ell \wedge Z \wedge E\tilde{\mathbf{P}}^\diamond$. Since X is $(\Sigma_n, 2\ell-1)$-equivalent to Y, X/Σ is $(2\ell-1)$-equivalent to $Y/\Sigma \simeq S^\ell \wedge Z \wedge B\tilde{\mathbf{P}}^\diamond$. Now apply Proposition 7.3 of **[2]** to get the result.

□

12.0.2. *4.3-4.4: m-equivalences.* If $\pi_k A = 0$ for $0 \le k \le m$, we say that A is *m-connected.* If $f : A \to B$ is a map of pointed spaces which induces an isomorphism on π_k for $0 \le k < m$ and is a surjection on π_m, we say that f is m-connected. If the induced map on π_m is also injective, we say that f is an *m-equivalence.* The following are some basic lemmas about this notion which were necessary in the proofs of stable equivalences in 4.3.

LEMMA 12.2. *Let A and B be pointed topological spaces, such that A is m-connected, for some nonnegative integer m. Then $A \wedge B$ is m-connected.*

PROOF. By assumption, $\pi_0(\Omega^k A) = 0$ for $0 \le k \le m$. Thus,

$$\pi_0(\Omega^k(A \wedge B)) = \pi_0(\Omega^k A \wedge \Omega^k B) = 0$$

Here, we have used the fact that $\mathrm{Map}_*(S^k, A \wedge B) = \mathrm{Map}_*(S^k, A) \wedge \mathrm{Map}_*(S^k, B)$ and that the smash product of a connected space with any pointed space, is itself connected. $\square$

LEMMA 12.3. *If $A \to X$ and $B \to Y$ are m-connected, then $A \wedge B \to X \wedge Y$ is m-connected.*

PROOF. Consider the composition $A \wedge B \to X \wedge B \to X \wedge Y$. We will show that the first map is m-connected. Then it will follow by a symmetry argument that the second also is, and clearly the composition of two m-connected maps is itself m-connected. To show a map is m-connected, it is equivalent to show that the cofiber is m-connected (by the long exact sequence in homotopy associated to a cofiber sequence), so let's compute that:

$$\mathrm{cof}(A \wedge B \to X \wedge B) \simeq \mathrm{cof}(A \to X) \wedge B$$

By assumption, $\mathrm{cof}(A \to X)$ is n-connected, so by the lemma above, so is $\mathrm{cof}(A \to X) \wedge B$, which completes the proof. $\square$

LEMMA 12.4. *If $X \to Y$ is m-connected, then $(\mathrm{Sp}^n/\mathrm{Sp}^{n-1})(X) \to (\mathrm{Sp}^n/\mathrm{Sp}^{n-1})(Y)$ is m-connected.*

PROOF. Clearly $X^{\wedge n} \to Y^{\wedge n}$ is m-connected on all fixed point spaces (as a map of Σ_n-spaces), and therefore, $X^{\wedge n}/\Sigma_n \to Y^{\wedge n}/\Sigma_n$ is m-connected. Similarly, $\mathrm{Sp}^{n-1}X \to \mathrm{Sp}^{n-1}Y$ is m-connected. Now, write down the map of long exact sequences associated to the cofiber sequence $\mathrm{Sp}^{n-1} \to \mathrm{Sp}^n \to \mathrm{Sp}^n/\mathrm{Sp}^{n-1}$.

$$\begin{tikzcd}
\pi_m \mathrm{Sp}^{n-1}X \arrow[r] \arrow[d] & \pi_m \mathrm{Sp}^n X \arrow[r] \arrow[d] & \pi_m \mathrm{Pr}_1^n X \arrow[r] \arrow[d, dashed] & \pi_{m-1}\mathrm{Sp}^{n-1}X \arrow[r] \arrow[d, "\simeq"] & \pi_{m-1}\mathrm{Sp}^n X \arrow[r] \arrow[d, "\simeq"] & \cdots \\
\pi_m \mathrm{Sp}^{n-1}X \arrow[r] & \pi_m \mathrm{Sp}^n X \arrow[r] & \pi_m \mathrm{Pr}_1^n X \arrow[r] & \pi_{m-1}\mathrm{Sp}^{n-1}X \arrow[r] & \pi_{m-1}\mathrm{Sp}^n X \arrow[r] & \cdots
\end{tikzcd}$$

A standard application of the 5-Lemma shows that $\pi_k \mathrm{Pr}_1^n X \to \pi_k \mathrm{Pr}_1^n Y$ is an isomorphism for $0 \le k \le m-1$, and a surjection for $k = m$ (the relevant part of the diagram for $k = m$ is shown above). $\square$

12.0.3. *4.6: Induction and Restriction for the Bruhat-Tits Building.* Let $\mathcal{S}_V$ denote the category of subspaces of V. Let $F : \mathcal{S}_V \to \mathrm{Top}_*$ be a functor equipped with isomorphisms $F(X) \wedge F(W) \xrightarrow{\cong} F(X \oplus W)$ (whenever X and W are transverse) and $S^0 \xrightarrow{\cong} F(0)$. Fix $W \subset V$, and let $P_W \subset \mathrm{GL}_V$ denote the parabolic subgroup associated with the flag $0 \subset W \subset V$. Then we wish to prove the following statement

PROPOSITION 12.5. *There is an equivalence of pointed P_W-spaces*

$$\mathrm{Ind}_{\mathrm{GL}_{W^\perp} \times GL_W}^{P_W} ((\mathbf{B}_{W^\perp}^{\diamond} \wedge F(W^\perp)) * (\mathbf{B}_W^{\diamond} \wedge F(W)) \to \mathbf{B}_V^{\diamond} \wedge F(V)$$

Here, $W^\perp$ is any subspace of V transverse to W.

LEMMA 12.6. *Let V be an $\mathbb{F}_p$-vector space of dimension n, and let $W \subset V$ have dimension k. Say that a maximal flag $[V_1 \subset \cdots \subset V_{n-1}]$ is W-*transverse *if $V_{n-k} \cap W = 0$. Then the top cohomology group $H^{n-2}(\mathbf{B}_V; \mathbb{Z})$ is spanned by the W-transverse flags.*

PROOF. Any maximal flag $\mathcal{F} = [V_1 \subset \cdots \subset V_{n-1}]$ gives rise to a sequence of $n-1$ numbers

$$d(\mathcal{F}) = (\dim(V_i \cap W))_{1 \le i \le n-1}$$

which we will call the *degree.* Lexicographically ordered, the minimum and maximum such a sequence can be are $(0, \ldots, 0, 1, 2, \ldots, k-1)$ and $(0, 1, 2, \ldots, k-1, k, k, \ldots, k)$ (the first case occurring precisely when $\mathcal{F}$ is W-transverse).

The top cohomology $H^{n-2}(\mathbf{B}_V; \mathbb{Z})$ is generated by the $\mathbb{Z}$-valued functions on the set of maximal flags (modulo relations coming from the submaximal flags). If $\mathcal{F}$ is a flag, we can denote by $\mathcal{F}$ the function which equals 1 on $\mathcal{F}$ and 0 on all other flags. We will then show that any maximal flag $\mathcal{F}$ which is not W-transverse is cohomologous to a sum of flags of (lexicographically) smaller degree. It will then follow by induction that any maximal flag is cohomologous to a sum of W-transverse ones.

Let $\mathcal{F} = [V_1 \subset \cdots \subset V_{n-1}]$ be a maximal flag which is not W-transverse. Then there exist $V_{i-1} \subset V_i \subset V_{i+1}$, with $i \in \{1, \ldots, n-2\}$ such that $\dim(V_{i-1} \cap W) = \ell - 1$ and $\dim(V_i \cap W) = \dim(V_{i+1} \cap W) = \ell$. It is clear that for all V_i' between V_{i-1} and V_{i+1}, $\dim(V_i' \cap W) = \ell - 1$, except when $V_i' = V_i$ in which case we get ℓ. Therefore, for each such $V_i' \neq V_i$,

$$d([V_1 \subset \cdots \subset V_i' \subset \cdots \subset V_{n-1}]) < d([V_1 \subset \cdots \subset V_i \subset \cdots \subset V_{n-1}])$$

The claim now follows from the simple observation that

$$\delta([V_1 \subset \cdots \subset V_{i-1} \subset V_{i+1} \subset \cdots \subset V_{n-1}]) = \sum_{V_{i-1} \subset V_i' \subset V_{i+1}} (-1)^{i-1} [V_1 \subset \cdots \subset V_i' \subset \cdots \subset V_{n-1}]$$

$\square$

Proof. (of 12.5) Let $\mathcal{C}_W$ denote the set of subspaces of V which are transverse to W. Then U_W acts freely and transitively on $\mathcal{C}_W$. P_W thus acts on $\bigvee_{X \in \mathcal{C}_W} (\mathbf{B}_X^\Diamond \wedge F(X)) * (\mathbf{B}_W^\Diamond \wedge F(W))$ as follows.

(1) P_W acts on $\mathbf{B}_W^\Diamond \wedge F(W)$, because the matrices in P_W fix W.
(2) Because P_W fixes W, it permutes the elements of $\mathcal{C}_W$. If we fit any one $X \in \mathcal{C}_W$, the matrices in P_W which fix X are $\mathrm{GL}_X \times \mathrm{GL}_W$.

Therefore, as a P_W-space,

$$\mathrm{Ind}_{\mathrm{GL}_{W^\perp} \times GL_W}^{P_W}((\mathbf{B}_{W^\perp}^\Diamond \wedge F(W^\perp)) * (\mathbf{B}_W^\Diamond \wedge F(W)) \simeq \bigvee_{X \in \mathcal{C}_W} (\mathbf{B}_X^\Diamond \wedge F(X)) * (\mathbf{B}_W^\Diamond \wedge F(W))$$

We will now show that there is an equivalence from the right hand side of this expression to $\mathbf{B}_V^\Diamond \wedge F(V)$. There is a P_W-equivalence

$$\bigvee_{X \in \mathcal{C}_W} (\mathbf{B}_X^\Diamond \wedge F(X)) * (\mathbf{B}_W^\Diamond \wedge F(W)) \simeq \left(\bigvee_{X \in \mathcal{C}_W} (\mathbf{B}_X^\Diamond * \mathbf{B}_W^\Diamond) \right) \wedge F(V)$$

where P_W acts simultaneously on the wedge sum and on $F(V)$. This is because $F(X) \wedge F(W) \simeq F(V)$ is a $(\mathrm{GL}_X \times \mathrm{GL}_W)$-equivalence. The result now follows from the case where F is the constant functor S^0, proven in [**7**]. We prove it here for completeness. We want a P_W-equivariant equivalence

$$\bigvee_{X \in \mathcal{C}_W} \mathbf{B}_X^\Diamond * \mathbf{B}_W^\Diamond \to \mathbf{B}_V^\Diamond$$

The maps $\mathbf{B}_X^\Diamond * \mathbf{B}_W^\Diamond \to \mathbf{B}_V^\Diamond$ arise on the level of simplices (with an appropriate dimension shift)

$$[X_1 \subsetneq \cdots \subsetneq X_i] * [W_1 \subsetneq \cdots \subsetneq W_j] \mapsto [X_1 \subsetneq \cdots X_i \subsetneq X \subsetneq X + W_1 \subsetneq \cdots \subsetneq X + W_j]$$

These maps are clearly P_W-equivariant. On maximal flags, this map is an inclusion into the set of W-transverse maximal flags in V. Therefore, since the top homology of $\mathbf{B}_V$ is spanned by the W-transverse maximal flags (by 12.6), this map is an isomorphism on the top cohomology groups. Because both sides have only one reduced cohomology group, it follows that this map is an equivalence.

□

12.0.4. *4.9: Lemmas about the Symmetric group.* The follow two lemmas are used in 4.9.

LEMMA 12.7. *Let $\sigma \in \Sigma_p$ be a p-cycle, and let $H \subset \Sigma_p$ be a nontransitive subgroup which is normalized by σ. Then H is the trivial group. Therefore, if $f : G \to \Sigma_p$ is a nontrivial homomorphism, then any subgroup of $G \times \Sigma_p$ which properly contains Γ_f has intersection with Σ_p transitive.*

PROOF. Without loss of generality, σ is the p-cycle sending $i \mapsto i+1 \pmod p$. Define an equivalence relation on $\{1,\ldots,p\}$ by $i \sim j$ if there is a permutation in H which sends i to j. Since H is normalized by σ, $i \sim j \iff i+k \sim j+k$ for all i,j,k (where all sums are taken modulo p). Therefore, if $i \sim i+k$ for some $k \neq 0$, we must have

$$i+k \sim i+2k \sim i+3k \sim \cdots \sim i+(p-1)k$$

and therefore there is only one equivalence class. This is impossible because H is nontransitive. Therefore, no two distinct indices are equivalent, and thus, H is the trivial group. This proves the first conclusion.

Now suppose that $f : G \to \Sigma_p$ is a nontrivial homomorphism, and let the generator $g \in G$ map to a p-cycle σ. Any subgroup $\Gamma \subset G \times \Sigma_p$ which contains Γ_f must, by the pigeonhole principle, contain two elements (g^i, π) and (g^i, π') for some $\pi, \pi' \in \Sigma_p$. Therefore, Γ intersects Σ_p nontrivially - call this intersection H. $(g,\sigma)H(g^{-1},\sigma^{-1}) \subset \Gamma$ because Γ is a group, and $\sigma H \sigma^{-1} = (g,\sigma)H(g^{-1},\sigma^{-1}) \subset \Sigma_p$ because G and Σ_p commute. Thus, $\sigma H \sigma^{-1} = H$, which implies H is transitive by the first part of the lemma. $\square$

LEMMA 12.8. *Let β denote the reduced standard representation of Σ_p (of dimension $p-1$), and let $\Gamma \subset G \times \Sigma_p$ be a subgroup. Then $(\rho_G \otimes \beta)^\Gamma = 0$ if and only if $\Gamma \cap \Sigma_p$ acts transitively on $\{1,2,\ldots,p\}$.*

PROOF. One direction is clear: if $\Gamma \cap \Sigma_p$ is transitive, then Γ contains an element $(1,\sigma)$ where $\sigma \in \Sigma_p$ is a p-cycle. Clearly $\beta^{\langle\sigma\rangle} = 0$ and thus $(\rho_G \otimes \beta)^\Gamma \subset (\rho_G \otimes \beta)^{\langle(1,\sigma)\rangle} = 0$.

So we must show that if $\Gamma \cap \Sigma_p$ is not transitive, then there is a nontrivial fixed vector. Let Γ' denote $\Gamma \cap \Sigma_p$. Then Γ' has a nontrivial fixed vector $w \in \beta$. For example, if Γ' is contained in a Young subgroup $\Sigma_i \times \Sigma_{p-i}$, then it fixes the vector $(p-i, p-i, \ldots, p-i, -i, -i, \ldots, -i)$ in the standard representation of Σ_p. This vector is contained within the copy of β in the standard representation (β is the subspace of vectors with sum of coordinates zero).

If $\Gamma = \Gamma'$, we are done, $v \otimes w$ is Γ-fixed for any $v \in \rho_G$. If not, then Γ is generated by Γ' along with an element (g, σ) where g is the generator of G. Clearly $\sigma^p \in \Gamma'$. Then, consider the vector

$$u = \sum_{i=0}^{p-1} g^i v \otimes \sigma^i w$$

where $v \in \rho_G$ is a vector upon which G acts freely. Clearly u is fixed by Γ' and by (g, σ), and it is nonzero because the elements $v, gv, g^2v, \ldots, g^{p-1}v$ are linearly independent. $\square$

CHAPTER 5

A decomposition of the Equivariant Dual Steenrod Algebra

This chapter is work towards producing an explicit decomposition of $M_{C_2}(k) \wedge H\underline{\mathbb{F}}_2$ into a wedge sum of suspensions of $H\underline{\mathbb{F}}_2$ by *representation spheres*. In 5.1, we define the category of chain complexes of permutation modules - $X \wedge H\underline{\mathbb{F}}_2 = \underline{\mathbb{F}}_2[X]$ lives in this category for any G-space X. In 5.2, we explicitly compute

$$\underline{\mathbb{F}}_2[B_{C_2}\Delta_1] \simeq \underline{\mathbb{F}}_2\{S^0, S^\sigma, S^{1+\sigma}, S^{1+2\sigma}, \ldots\}$$

in this category of chain complexes. Then, we can identify the summand $\underline{\mathbb{F}}_2[\epsilon_k B_{C_2}\Delta_k] \subset \underline{\mathbb{F}}_2[B_{C_2}\Delta_k] \simeq \underline{\mathbb{F}}_2[B_{C_2}\Delta_1]^{\otimes k}$ by a computation on the level of *geometric fixed points* - proving that this can be done is the content of 5.3. This reduces the computation of $\underline{\mathbb{F}}_2[M_{C_2}(k)]$ to an ordinary algebraic computation with $\mathbb{F}_2$-vector spaces:

THEOREM 0.9. *4.1*

$\underline{\mathbb{F}}_2[\epsilon_k B_{C_2}\Delta_k]$ *can be decomposed by computing* $\epsilon_k S_k$*, where* S_k *is the* $RO(G)$*-graded ring*

$$S_k := \mathrm{Sym}^*(\mathbb{F}_2^k) \otimes \Lambda^*(\mathbb{F}_2^k) \simeq \mathbb{F}_2[x_1, \ldots, x_k] \otimes \Lambda[s_1, \ldots, s_k]$$

with $\deg(x_i) = \rho_{C_2}$ *and* $\deg(s_i) = \sigma$*, and where* GL_k *acts in the natural way on each part.*

This Steinberg summand should have a computable basis analogous to that in (Chapter 2, 8.3), although we haven't yet been able to explicitly find this basis. In 5.5, we discuss the obstruction to generalizing the explicit computations of section 2.9, and in 5.6, we discuss the difficulties of generalizing section 2.10.

1. Chain complexes of Permutation Modules

Recall that a finite dimensional $\mathbb{F}_2[C_2]$-module is called a *permutation module* if it is of the form $\mathbb{F}_2\{T\}$ for some finite C_2-set T. We define the *category of mod 2 permutation modules*, denoted Perm_{C_2}, to be generated by the full image of the functor

$$C_2\mathrm{Set} \xrightarrow{\mathbb{F}_2[-]} C_2\mathrm{Mod}$$

Any finite C_2-set is a disjoint union of trivial orbits (C_2/C_2) and free orbits $(C_2/1)$, and therefore, any permutation module[1] is a direct sum of copies of the indecomposables $\mathbb{F}_2$ and $\mathbb{F}_2[C_2]$. Thus, if $M \simeq \mathbb{F}_2\{T\}$ is a permutation module, we have a decomposition $M \simeq M_{\mathrm{free}} \oplus \Phi^{C_2} M$ with

$$\Phi^{C_2} M := \mathbb{F}_2\{T^{C_2}\} \qquad M_{\mathrm{free}} := \mathbb{F}_2\{T_{\mathrm{free}}\}$$

where T_2^C, T_{free} are the free and fixed parts, respectively, of the C_2-set T. Moreover, in the category of permutation modules, any morphism is a sum of morphisms induced from C_2-set maps: for example, there are no morphisms $\mathbb{F}_2 \to \mathbb{F}_2[C_2]$. Therefore, if $M \to N$ is a morphism of permutation modules, it induces maps

$$\begin{tikzcd}
M_{\mathrm{free}} \arrow[d] \arrow[dr] & \Phi^{C_2} M \arrow[d] \\
N_{\mathrm{free}} & \Phi^{C_2} N
\end{tikzcd}$$

Definition 1.1. *The category* ChPerm_{C_2} *is the category of* unbounded chain complexes *of permutation modules, where all differentials and maps between chain complexes come from the category of permutation modules. Any chain complex of permutation modules V has a corresponding decomposition*

$$V \simeq \{V_{\mathrm{free}} \xrightarrow{d} \Phi^{C_2} V\}$$

[1]All permutation modules we consider will be mod 2 permutation modules, so we just call them permutation modules for simplicity.

where $(-)_{\text{free}}$ *and* $\Phi^{C_2}(-)$ *are functors*

$$(-)_{\text{free}}, \Phi^{C_2}(-) : \mathrm{ChPerm}_{C_2} \to \mathrm{Ch}_{\mathbb{F}_2}$$

arising from the corresponding functors on Perm_{C_2}.

For example, if X is a C_2-space, then $\underline{\mathbb{F}_2}[X]$ is a complex of permutation modules. We will be interested in the question of how to decompose a complex of permutation modules into indecomposables.[2] One particular type of indecomposable that we will be interested in is the set of *representation spheres.*

DEFINITION 1.2. *For any* $a, b \geq 0$, *let*

$$\underline{\mathbb{F}_2}[S^{a+b\sigma}] := \{\underbrace{\mathbb{F}_2[C_2] \to \cdots \to \mathbb{F}_2[C_2]}_{b \text{ copies}} \to \mathbb{F}_2\}$$

where the $\mathbb{F}_2$ *is in degree* a. *This is called a* representation sphere of length b.

$S^{a+b\sigma}$ is the C_2-simplicial set associated to the one-point compactification of a times the trivial representation plus b times the sign representation - this representation sphere has an equivariant cell decomposition with a trivial bottom cell in degree a and free cells in degrees $a+1, a+2, \ldots, a+b$, and so the above is sensible notation. We will want to consider representation spheres appearing as subcomplexes of a complex of permutation modules V.

2. A subcomplex of $\underline{\mathbb{F}_2}[B_{C_2}\Delta_1]$

In this section, our goal is to prove

PROPOSITION 2.1. $\underline{\mathbb{F}_2}[B_{C_2}\Delta_1]$ *has a decomposition*

$$\underline{\mathbb{F}_2}[B_{C_2}\Delta_1] \simeq \left(\bigoplus_{m=0}^{\infty} Y_m\right) \oplus Z$$

[2]By the Krull-Schmidt theorem, any two decompositions of a permutation complex into indecomposables are equivalent.

where Z is a contractible subcomplex and $Y_m \simeq \underline{\mathbb{F}}_2\{S^{m+m\sigma}, S^{m+(m+1)\sigma}\}$ for each $m \geq 0$.

In particular, this implies that

$$\underline{\mathbb{F}}_2[B_{C_2}\Delta_k] \simeq \left(\bigoplus_{m_1,\ldots,m_k \geq 0} Y_{m_1} \otimes \cdots \otimes Y_{m_k} \right) \oplus (\text{contractible complex})$$

which we will use in the next section. We first construct linearly independent subcomplexes $Y_k \simeq \underline{\mathbb{F}}_2[S^{k+k\sigma}, S^{k+(k+1)\sigma}]$ for $k \geq 0$. Then we show that there is a contractible complement Z.

Write $\Delta = \Delta_1$ to mean the group $\mathbb{Z}/2$. Our explicit cell decomposition for $E_{C_2}\Lambda$, for any group Λ, allows us to analyze $\underline{\mathbb{F}}_2[B_{C_2}\Delta]$. Recall that $E_{C_2}\Delta$ has n-simplices

$$(E_{C_2}\Delta)_n = \Delta^{\times n} \sqcup (\Delta^{\times(n-1)} \times \Delta_\Gamma) \sqcup \cdots \sqcup \Delta_\Gamma^{\times n}$$

where Δ_Γ is a copy of Δ with C_2 acting via the nontrivial graph subgroup of $G \times \Delta$. Therefore, $B_{C_2}\Delta$ has n-simplices

$$(B_{C_2}\Delta)_n = \Delta^{\times(n-1)} \sqcup (\Delta^{\times(n-2)} \times \Delta_\Gamma) \sqcup (\Delta^{\times(n-3)} \times \Delta_\Gamma \times {}_\Gamma\Delta_\Gamma) \sqcup \cdots \sqcup ({}_\Gamma\Delta_\Gamma)^{\times(n-1)}$$

Here, Δ_Γ should be thought of as Δ with a nontrivial *right* action of C_2, and ${}_\Gamma\Delta_\Gamma$ as a copy of Δ with C_2 acting simultaneously on the left and right (therefore, acting trivially). The face maps arise in the natural way - for example, on 2-simplices, the maps look as shown.

$$\begin{tikzcd}
\Delta \times \Delta \arrow[dr, shift left] \arrow[dr] \arrow[dr, shift right] & \Delta \times \Delta_\Gamma \arrow[d] \arrow[dr, shift left] \arrow[dr, shift right] & & \Delta_\Gamma \times {}_\Gamma\Delta_\Gamma \arrow[dl, shift left] \arrow[dl, shift right] \arrow[d] & {}_\Gamma\Delta_\Gamma \times {}_\Gamma\Delta_\Gamma \arrow[dl, shift left] \arrow[dl] \arrow[dl, shift right] \\
& \Delta & \Delta_\Gamma & {}_\Gamma\Delta_\Gamma &
\end{tikzcd}$$

Let $1, \tau$ denote the trivial and nontrivial elements, respectively of Δ. For every pair $i, j \geq 0$, define elements

$$x_{i,j} = (1+\tau)^{\otimes i} \otimes 1 \otimes (1+\tau)^{\otimes j} \in \mathbb{F}_2[\Delta^{\times i} \times \Delta_\Gamma \times {}_\Gamma\Delta_\Gamma^{\times j}]$$

$$x_{i,-1} = z_i = (1+\tau)^{\otimes i} \in \Delta^{\times i} \qquad x_{-1,j} = w_j = (1+\tau)^{\otimes j} \in \mathbb{F}_2[{}_\Gamma\Delta_\Gamma^{\times j}]$$

Proposition 2.2. *The z_i's and w_j's generate the $\mathbb{F}_2$-homology of $(B_{C_2}\Delta)^{C_2} \simeq B\Delta \vee B({}_\Gamma\Delta_\Gamma)$.*

Proof. This is an immediate consequence of 6.2. □

Now for $0 \leq \ell \leq m-1$, define elements

$$a_{m,\ell} = \sum_{j=0}^{\ell} \binom{\ell}{j} x_{m-j,m-1-(\ell-j)} \qquad b_{m,\ell} = \sum_{j=0}^{\ell} \binom{\ell}{j} x_{m-1-(\ell-j),m-j}$$

and for $\ell = m$,

$$a_{m,m} = z_m + (1+\tau)\sum_{j=1}^{m} \binom{m}{j} x_{m-j,j-1} \qquad b_{m,m} = w_m + (1+\tau)\sum_{j=1}^{m} \binom{m}{j} x_{j-1,m-j}$$

One can then easily check the following equations.

$$dx_{m,m} = (1+\tau)(x_{m,m-1} + x_{m-1,m}) = (1+\tau)(a_{m,0} + b_{m,0})$$

$$da_{m,\ell} = (1+\tau)a_{k,\ell+1} \quad db_{m,\ell} = (1+\tau)b_{m,\ell+1} \quad \text{if} \quad 0 \leq \ell \leq m-2$$

$$da_{m,m-1} = a_{m,m} \qquad db_{m,m-1} = b_{m,m}$$

Then $\underline{\mathbb{F}}_2[B_{C_2}\Delta]$ has the folllowing subcomplex, which we denote Y_m.[3]

$$\begin{tikzcd}
\mathbb{F}_2\{x_{m,m}, \tau x_{m,m}\} \arrow[d] \arrow[dr] & & \simeq & \mathbb{F}_2[C_2] \arrow[d] \arrow[dr] & \\
\mathbb{F}_2\{a_{m,0}, \tau a_{m,0}\} \arrow[d] & \mathbb{F}_2\{b_{m,0}, \tau b_{m,0}\} \arrow[d] & & \mathbb{F}_2[C_2] \arrow[d] & \mathbb{F}_2[C_2] \arrow[d] \\
\mathbb{F}_2\{a_{m,1}, \tau a_{m,1}\} \arrow[d] & \mathbb{F}_2\{b_{m,1}, \tau b_{m,1}\} \arrow[d] & & \mathbb{F}_2[C_2] \arrow[d] & \mathbb{F}_2[C_2] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] \\
\mathbb{F}_2\{a_{m,m-1}, \tau a_{m,m-1}\} \arrow[d] & \mathbb{F}_2\{b_{m,m-1}, \tau b_{m,m-1}\} \arrow[d] & & \mathbb{F}_2[C_2] \arrow[d] & \mathbb{F}_2[C_2] \arrow[d] \\
\mathbb{F}_2\{a_{m,m}\} & \mathbb{F}_2\{b_{m,m}\} & & \mathbb{F}_2 & \mathbb{F}_2
\end{tikzcd}$$

Every simplex of Y_m has a 'highest term' which has at least m parts in Δ (tensored together) *or* at least m parts in ${}_\Gamma\Delta_\Gamma$ (tensored together), and therefore, the simplices of the various Y_m's are linearly independent. Thus, $Y = \bigoplus_{m=0}^{\infty} Y_m$ is a subcomplex of $\underline{\mathbb{F}}_2[B_{C_2}\Delta]$. It is clear that $Y_m \simeq \underline{\mathbb{F}}_2\{S^{m+m\sigma}, S^{m+(m+1)\sigma}\}$, and so $Y \simeq \underline{\mathbb{F}}_2\{S^0, S^\sigma, S^{1+\sigma}, S^{1+2\sigma}, S^{2+2\sigma}, \ldots\}$

PROPOSITION 2.3. *$\underline{\mathbb{F}}_2[B_{C_2}\Delta] \simeq Y \oplus Z$ for a contractible subcomplex $Z \subset \underline{\mathbb{F}}_2[B_{C_2}\Delta]$, thus proving 2.1. In particular,*

$$\underline{\mathbb{F}}_2[B_{C_2}\Delta] \simeq \underline{\mathbb{F}}_2\{S^0, S^\sigma, S^{1+\sigma}, S^{1+2\sigma}, S^{2+2\sigma}, \ldots\}$$

PROOF. Z is defined by the following cells:

[3] The isomorphism is because C_2 acts freely on $x_{m,m}$ and on every $a_{m,\ell}$ and $b_{m,\ell}$ when $0 \leq \ell \leq m-1$, and C_2 acts trivially on $a_{m,m}$ and $b_{m,m}$. Here, $a_{m,m}$ and $b_{m,m}$ are in degree m, so Y_m has its bottom cells in degree m and its top cell in degree $2m+1$.

(1) Consider a basis for $\mathbb{F}_2[\Delta^{\times m}]$ (where $m \geq 0$) consisting of m-fold tensor products, where each term is 1 or $1+\tau$. Then in this component, Z is spanned by all monomials *except* the forbidden monomial $(1+\tau)^{\otimes m}$ (so Z has dimension $2^m - 1$). Z contains the same monomials in $\mathbb{F}_2[{}_\Gamma\Delta_\Gamma^{\times m}]$.

(2) Similarly, consider a basis for $\mathbb{F}_2[\Delta^{\times i} \times \Delta_\Gamma \times {}_\Gamma\Delta_\Gamma^{\times j}]$ of 2^{i+j+1} monomials formed out of 1 and $1+\tau$. Then in this component, Z is spanned by all monomials *except* $(1+\tau)^{\otimes i} \otimes 1 \otimes (1+\tau)^{\otimes j}$ and $(1+\tau)^{\otimes i} \otimes (1+\tau) \otimes (1+\tau)^{\otimes j}$. That is, in these components, Z has dimension $2^{i+j+1} - 2$, containing all monomials except these two forbidden ones.

Z is disjoint from Y because every simplex of Y contains a component with one of these forbidden monomials. It is clear by dimension reasons that $Z_n \oplus Y_n \simeq \underline{\mathbb{F}_2}[(B_{C_2}\Delta)_n]$, so now in order to prove that $\underline{\mathbb{F}_2}[B_{C_2}\Delta] \simeq Y \oplus Z$, it suffices for us to prove that Z is a subcomplex (i.e., that it is preserved by the action of C_2 and that the differential of any simplex in Z still lies in Z). Z is clearly preserved by the action of C_2. To prove the second requirement, it suffices to show that the differential of any sum of non-forbidden monomials, is again a sum of non-forbidden monomials.

(1) For the forbidden monomial $(1+\tau)^{\otimes m} \in \mathbb{F}_2[\Delta^{\times m}]$, the only non-forbidden monomials with this as a face lie in $\mathbb{F}_2[\Delta^{\times(m+1)}]$, and we have already proven that $(1+\tau)\otimes m$ is not in the image of the differential from $\mathbb{F}_2[\Delta^{\times(m+1)}]$. An analogous argument holds for $(1+\tau)^{\otimes m} \in \mathbb{F}_2[{}_\Gamma\Delta_\Gamma^{\times m}]$.

(2) For the forbidden monomial $(1+\tau)^{\otimes i} \otimes 1 \otimes (1+\tau)^{\otimes j} \in \mathbb{F}_2[\Delta^{\times i} \times \Delta_\Gamma \times {}_\Gamma\Delta_\Gamma^{\times j}]$, the only monomials with this as a face are

$$(1+\tau)^{\otimes i_1} \otimes 1 \otimes (1+\tau)^{\otimes i_2} \otimes 1 \otimes (1+\tau)^{\otimes j} \in \mathbb{F}_2[\Delta^{\times(i+1)} \times \Delta_\Gamma \times {}_\Gamma\Delta_\Gamma^{\times j}]$$

with $i_1 + i_2 = i$, or

$$(1+\tau)^{\otimes i} \otimes 1 \otimes (1+\tau)^{\otimes j_1} \otimes 1 \otimes (1+\tau)^{\otimes j_2} \in \mathbb{F}_2[\Delta^{\times i} \times \Delta_\Gamma \times {}_\Gamma\Delta_\Gamma^{\times(j+1)}]$$

with $j_1+j_2=j$. Each of these has $(1+\tau)^{\otimes i}\otimes 1\otimes(1+\tau)^{\otimes j}$ as a face precisely twice, so when we apply the differential, these terms cancel out.

Thus, Z is indeed a subcomplex, and thus $\underline{\mathbb{F}}_2[B_{C_2}\Delta]\simeq Y\oplus Z$. All that remains now is to show that Z is contractible. To show this, it suffices to show that $\underline{\mathbb{F}}_2[B_{C_2}\Delta]$ and Y have the same Mackey functor homotopy groups. From the definition of Y, $\pi_*^{(e)}(Y)$ is one-dimensional in every degree and $\pi_*^{C_2}(Y)$ is two-dimensional in every degree. The same holds for $\underline{\mathbb{F}}_2[B_{C_2}\Delta]$, because $(B_{C_2}\Delta)^{(e)}\simeq B\Delta$ and $(B_{C_2}\Delta)^{C_2}\simeq B\Delta\vee B\Delta$. □

3. Computing lengths

In this section, we want to prove roughly that

THEOREM 3.1. *One can obtain a decomposition of $\epsilon_k\underline{\mathbb{F}}_2[B_{C_2}\Delta_k]$ into representation spheres by calculating*

$$\epsilon_k\mathbb{F}_2[(B_{C_2}\Delta_k)^G]\simeq\epsilon_k\mathbb{F}_2[x_1,\dots,x_k][\Delta_k]$$

$$\simeq\epsilon_k\mathbb{F}_2[x_1,\dots,x_k,t_1,\dots,t_k]/(t_1^2=\cdots=t_k^2=1)$$

where GL_k *acts in the natural way on the linear forms* $\langle x_1,\dots,x_k\rangle$ *and simultaneously in the natural way on* $\Delta_k=\{t_1^{d_1}\cdots t_k^{d_k}:d_i\in\{0,1\}\}$*. Here there is an* $RO(G)$*-grading:* $\deg(x_i)=\rho_{C_2}$ *and* $\deg(1+t_i)=\sigma$*. Each monomial* $x_1^{i_1}\cdots x_k^{i_k}(1+t_1)^{d_1}\cdots(1+t_k)^{d_k}$ *thus corresponds to a representation sphere* $\underline{\mathbb{F}}_2[S^{(\sum i_j)+(\sum(i_j+d_j))\sigma}]$ *in* $\underline{\mathbb{F}}_2[B_{C_2}\Delta_k]$*.*

We want to show that we can compute the representation sphere summands which appear in $\epsilon_k\underline{\mathbb{F}}_2[B_{C_2}\Delta_k]$ by instead analyzing the Steinberg summand of the fixed points $\epsilon_k\mathbb{F}_2[(B_{C_2}\Delta_k)^G]$. We will first develop this argument for a general complex $V\in\mathrm{ChPerm}_{C_2}$, and then apply the argument to the case where $V=\underline{\mathbb{F}}_2[B_{C_2}\Delta_k]$.

Consider a decomposition of some $V\in\mathrm{ChPerm}_{C_2}$ into a direct sum of indecomposables. If V carries an action of GL_k (which acts by maps of permutation modules, and therefore respects 1.1), then we have a splitting $V \simeq \epsilon_k V \oplus \epsilon_k^{\perp} V$.[4] By the Krull-Schmidt theorem, a copy of each indecomposable of V lies in either $\epsilon_k V$ or $\epsilon_k^{\perp} V$. We will develop a method to determine which representation spheres lie in which summand, showing that this can be deduced from $\epsilon_k \Phi^{C_2} V$ and $\epsilon_k^{\perp} \Phi^{C_2} V$.

Definition 3.2. *Call a representation sphere $\mathbb{F}_2[S^{i+j\sigma}]$ in V a* representation sphere summand *if it has a complement $V \simeq W \oplus \mathbb{F}_2[S^{i+j\sigma}]$.*

We need the following lemma.

Lemma 3.3. *Let $M \subset V$ be the submodule generated by the bottom cells of the representation spheres in the given splitting of V. Then $\Phi^{C_2} : M \to \Phi^{C_2} M$ is an isomorphism.*

The lemma above follows immediately from the following two propositions.

Proposition 3.4. *Let V be a complex of permutation modules, and let $\mathbb{F}_2[S^{i+j\sigma}] \subset V$ be such that $V \simeq W \oplus \mathbb{F}_2[S^{i+j\sigma}]$. Then $\Phi^G \mathbb{F}_2[S^{i+j\sigma}]$ is nontrivial in degree i.*

Proof. Let x be the bottom (trivial) cell of $\mathbb{F}_2[S^{i+j\sigma}]$, and suppose that $\Phi^G x = 0$. Then $x \in V_{\text{free}}^{C_2}$, and so there is some $y \in V_{\text{free}}$ such that $(1+\tau)y = x$. Write $y = (y_1, y_2)$ where y_1 is the component in W and y_2 is the component in $\mathbb{F}_2[S^{i+j\sigma}]$. Since $\mathbb{F}_2[S^{i+j\sigma}]$ has only one cell in this dimension, either $y_2 = x$ or $y_2 = 0$: either way, $(1+\tau)y = ((1+\tau)y_1, 0)$, which is a contradiction! Thus, $\Phi^{C_2} x \neq 0$. $\square$

Proposition 3.5. *Fix a splitting of V into indecomposables. Any homogeneous element $x \in V_i$ which is a linear combination of the bottom cells of representation sphere in the given splitting of V, is itself the bottom cell in a representation sphere summand. Hence, $\Phi^{C_2} x \neq 0$.*

[4]Remember that $\epsilon_k^{\perp} = 1 - \epsilon_k$ is an idempotent of $\mathbb{F}_p[\mathrm{GL}_k]$.

Proof. x is a sum of the bottom cells of representation spheres of various lengths: let $S \simeq \mathbb{F}_2[S^{i+j\sigma}]$ be the one of shortest length (i.e. with j minimal), and suppose $V \simeq W \oplus S$. Clearly, x itself is the bottom cell in a representation sphere of length j: call this representation sphere S'. Then it is easy to see that $V \simeq W \oplus S'$. $\square$

We make the following definition to encode the *lengths* of the representation spheres determined by elements in M or $\Phi^{C_2}M$.

Definition 3.6. *Let $y \in M$ be homogeneous. Then we define* $\operatorname{len}_V(y)$ *to be the largest number b such that y is the bottom cell in a representation sphere summand of length b. (When $y = 0$, we define* $\operatorname{len}_V(x) = \infty$.*)*

Let $x \in \Phi^{C_2}M$ be homogeneous. Then we define $\ell_V(x)$ to be the largest value of $\operatorname{len}_V(y)$ *taken over all homogeneous y such that $\Phi^{C_2}y = x$. We call this the* length *of x.*

Proposition 3.7. *For any b, the set of homogeneous $y \in V^{C_2}$ such that* $\operatorname{len}_V(y) \geq b$ *is a subspace, which we denote by $M(b)$. In particular, we have a descending sequence*

$$M = M(0) \supset M(1) \supset M(2) \supset \cdots$$

Similarly, the set of homogeneous $x \in \Phi^{C_2}M$ such that $\ell_V(x) \geq b$ is a subspace, which we denote by M^b. In particular, $M(b) \xrightarrow{\simeq} \Phi^{C_2}M(b) = M^b$.

Proof. One can easily check that $\operatorname{len}_V(-)$ satisfies properties analogous to those of a non-Archimedean valuation, namely

(1) If $\operatorname{len}_V(y) < \operatorname{len}_V(y')$, then $\operatorname{len}_V(y + y') = \operatorname{len}_V(y)$.
(2) If $\operatorname{len}_V(y) = \operatorname{len}_V(y')$, then $\operatorname{len}_V(y + y') \geq \operatorname{len}_V(y)$.

The first part of the proposition follows immediately from these properties. It's true by definition that $M^b = \Phi^{C_2}M(b)$, so the M^b's form subspaces as well. $\square$

Now specialize to $V = \underline{\mathbb{F}}_2[B_{C_2}\Delta_k]$, and consider splittings of $\epsilon_k V$ and $\epsilon_k^{\perp} V$ into indecomposables - this determines a splitting of V into indecomposables, where some of the summands are preserved by ϵ_k and some are killed by ϵ_k.

Let M be the $\mathbb{F}_2$-vector space spanned by the bottom cells of the representation sphere summands of V. By 3.3, $\Phi^{C_2} : M \to \Phi^{C_2} M$ is an isomorphism, and so $\Phi^{C_2} M$ generates the homology of $\Phi^{C_2} V$ (because we explicitly computed in 2.1 that the bottom cells of the representation spheres generate the homology of the fixed points of $B_{C_2}\Delta_k$). Therefore, $\Phi^{C_2} M \hookrightarrow \Phi^{C_2} V$ is an equivalence on homology (where $\Phi^{C_2} M$ has no differentials).

Now $\epsilon_k M$ is the vector space spanned by the bottom cells of $\epsilon_k V$. Since ϵ_k is a projective idempotent, it's then easy to see we have a homology equivalence

$$\epsilon_k M \xrightarrow{\simeq} \epsilon_k \Phi^{C_2} M \to \epsilon_k \Phi^{C_2} V$$

So we can compute which representation sphere summands have their bottom cell in $\epsilon_k M$, by computing the Steinberg summand of the homology of $\Phi^{C_2} V$. So now we will compute the GL_k action on the homology of $\Phi^{C_2} V$, and then use this to compute how many representation sphere summands of the form $\underline{\mathbb{F}}_2[S^{a+b\sigma}]$ $(a, b \geq 0)$ live in $\epsilon_k V$.

Let $H_k \subset \Phi^{C_2} \underline{\mathbb{F}}_2[B_{C_2}\Delta_k] \simeq \mathbb{F}_2[B_{C_2}\Delta_k^{C_2}]$ denote the subcomplex spanned by the homology generators (so that $H_k \simeq \mathbb{F}_2[B_{C_2}\Delta_k^{C_2}]$). For example, when $k = 1$, H_k is spanned by $\{a_{i,i}, b_{i,i}\}_{i \geq 0}$. The ring structure $B_{C_2}\Delta_1 \wedge B_{C_2}\Delta_1 \to B_{C_2}\Delta_1$ induces a ring structure on the fixed points, and this gives a ring structure on H_k. We can use our computation from the nonequivariant case to compute the ring structure on this complex.

Proposition 3.8. *$H_1 \simeq \mathbb{F}_2[x, t]/(t^2 - 1)$, where x is the 1-cell of the trivial copy of $B\Delta_1$, and t is the 0-cell of the nontrivial copy of $B\Delta_1$. (The trivial copy and nontrivial copy corresponding to the zero and nonzero element of Δ_1, respectively.)*

Proof. The product map on $\left(\bigvee_{\Delta_1} B\Delta_1\right) \simeq B\Delta_1 \vee B\Delta_1$ is assembled from the usual product map on $B\Delta_1$, combined with the addition of elements in Δ_1. The proposition is now more or less obvious. Explicitly, x^i corresponds to the simplex $a_{i,i}$, and $x^i t$ corresponds to the simplex $b_{i,i}$. $\square$

Corollary 3.9.

$$\Phi^{C_2}\underline{\mathbb{F}}_2[B_{C_2}\Delta_k] \simeq H_k \simeq \mathbb{F}_2[x_1, \dots, x_k][\Delta_k]$$

$$\simeq \mathbb{F}_2[x_1, \dots, x_k, t_1, \dots, t_k]/(t_1^2 = \cdots = t_k^2 = 1)$$

GL_k acts in the natural way on the linear forms $\langle x_1, \dots, x_k\rangle$, and simultaneously in the natural way on $\Delta_k = \{t_1^{d_1} \cdots t_k^{d_k} : d_i \in \{0,1\}\}$.

Proposition 3.10. *In $H_k \simeq \mathbb{F}_2[x_1, \dots, x_k][\Delta_k]$,*

$$\ell_V(x_1^{i_1} \cdots x_k^{i_k}(1+t_1)^{d_1} \cdots (1+t_k)^{d_k}) = \sum_{j=1}^{k}(i_j + d_j)$$

where each $d_j \in \{0,1\}$ and $i_j \geq 0$. Here

- *$x_j^i = (1+\tau)^{\otimes i} \in \mathbb{F}_2[\Delta_1^{\times i}]$ is the i-simplex coming from x^i under the j-th copy $f_j : B_{C_2}\Delta_1 \hookrightarrow B_{C_2}\Delta_1^{\wedge k} \simeq B_{C_2}\Delta_k$.*
- *$1, t_j$ are the two 0-cells of $B_{C_2}\Delta_1 \xhookrightarrow{f_j} B_{C_2}\Delta_k$.*

Any homogeneous $z \in H_k$ can be written uniquely as a linear combination of these monomials. Then $\ell_V(z)$ is the minimal value of ℓ_V among these monomials.

Proof. When $k = 1$, this is clear because x^i corresponds to $a_{i,i}$ and $x^i(1+t)$ corresponds to $a_{i,i} + b_{i,i}$: these correspond to the bottom cells in the two summands of Y_i, defined in the previous section, and have length i and $i+1$. For general k, this holds because if we have $z_1, \dots, z_k \in \Phi^{C_2}V$, then $z_1 \otimes \cdots \otimes z_k \in \Phi^{C_2}(V^{\otimes k}) = (\Phi^{C_2}V)^{\otimes k}$ and

$$\ell_{V^{\otimes k}}(z_1 \otimes \cdots \otimes z_k) = \ell_V(z_1) \cdots \ell_V(z_k)$$

$\square$

4. Open Question: Calculating $\epsilon_k\underline{\mathbb{F}_2}[B_{C_2}\Delta_k]$

Let $H_k = \mathbb{F}_2[(B_{C_2}\Delta_k)^{C_2}] \simeq \mathbb{F}_2[x_1,\ldots,x_k][\Delta_k]$, and define the ideal $I \subset H_k$ by $I = \langle 1+t_1,\ldots,1+t_k\rangle$. Clearly GL_k maps $I^d \to I^d$ for each $d \geq 0$. Thus, GL_k preserves the filtration

$$H_k = I^0 \supset I^1 \supset I^2 \supset \cdots \supset I^k \supset I^{k+1} = 0$$

Since ϵ_k is a projective idempotent, it preserves exact sequences and thus $\epsilon_k I^d/\epsilon_k I^{d+1} = \epsilon_k(I^d/I^{d+1})$.

Our goal is to calculate, for each ℓ, the length ℓ part of $\epsilon_k H_k$. These form a descending filtration (as ℓ increases). The upshot of the above discussion is that we may instead calculate the same thing for the Steinberg summand of the *associated graded ring* $S_k = \bigoplus_{d=0}^{k} I^d/I^{d+1}$. It is not hard to see that this is precisely the ring

$$S_k \simeq \mathrm{Sym}^*\mathbb{F}_2^k \otimes \Lambda^*\mathbb{F}_2^k \simeq \mathbb{F}_2[x_1,\ldots,x_k]\otimes\Lambda[s_1,\ldots,s_k]$$

where s_i is the image of $1+t_i$ in I/I^2, for $i = 1,\ldots,k$. Therefore, we have the following (possibly simpler) computation:

THEOREM 4.1. *$\underline{\mathbb{F}}_2[\epsilon_k B_{C_2}\Delta_k]$ can be decomposed by computing $\epsilon_k S_k$, where S_k is the $RO(G)$-graded ring*

$$S_k := \mathrm{Sym}^*(\mathbb{F}_2^k)\otimes\Lambda^*(\mathbb{F}_2^k) \simeq \mathbb{F}_2[x_1,\ldots,x_k]\otimes\Lambda[s_1,\ldots,s_k]$$

with $\deg(x_i) = \rho_{C_2}$ and $\deg(s_i) = \sigma$, and where GL_k acts in the natural way on each part.

QUESTION 4.2. *Compute a homogeneous basis for the Steinberg summand of S_k.*

This appears to be a complicated combinatorics question, but we hope it is possible to appropriately generalize the methods of [**31**] (see Chapter 2, 8.3). For example, the Steinberg summand of the parts $\mathbb{F}_2[x_1, \ldots, x_k] \otimes 1$ and $\mathbb{F}_2[x_1, \ldots, x_k] \otimes (s_1 \ldots s_k)$ each have a basis corresponding to admissible sequences of length k in $\mathcal{A}^*$. Along the lines of generalizing the proof of (Chapter 2, 8.3), we ask the following question.

QUESTION 4.3. *Can we produce an appropriate equivariant generalization of the Steenrod squares so that $\epsilon_k S_k$ has a basis in correspondence with the equivariant Steenrod squares of 'length k'?*

For example, there is an *equivariant Bockstein*, coming from the sequence

$$H\underline{\mathbb{Z}/2} \longrightarrow H\underline{\mathbb{Z}/4} \longrightarrow H\underline{\mathbb{Z}/2} \xrightarrow{\overline{\beta}} \Sigma H\underline{\mathbb{Z}/2}$$

5. Open Question: Computing the Ring Structure

QUESTION 5.1. *Can the explicit computations in Section 2.9 be methodically generalized to the equivariant setting?*

In the nonequivariant setting, every map $S^i \to S^j \wedge H\mathbb{F}_2$ with $i \neq j$ is zero. This allows us to explicitly compute relations among the elements of $D(k) \wedge H\mathbb{F}_2$ using the product structure, as done in Section 2.9. However, in the equivariant setting, there are nontrivial maps a_σ and u_σ between representation spheres in different dimensions.

DEFINITION 5.2. *There exist nontrivial maps*

$$a_\sigma : S^0 \to S^\sigma \wedge H\underline{\mathbb{F}}_2 \qquad\qquad u_\sigma : S^1 \to S^\sigma \wedge H\underline{\mathbb{F}}_2$$

*a_σ comes from the inclusion of the bottom cell $S^0 \to S^\sigma$, and u_σ comes from the generator of $C_1^{\mathrm{cell}}(S^\sigma, \underline{\mathbb{F}}_2)$ (see [**18**] example 3.10).*

For example, the splitting

$$\begin{tikzcd} \underline{\mathbb{F}}_2[D_{C_2}(1)] \arrow[r, shift right] & \underline{\mathbb{F}}_2[\Sigma M_{C_2}(1)] \arrow[l, dashed, shift right, "t_1"'] \end{tikzcd}$$

allows us to lift the module generators $\Sigma(x^iT^d)$ to elements $c_{i,d} = \underline{\mathbb{F}}_2[S^{1+i\rho_{C_2}+d\sigma}] \subset \underline{\mathbb{F}}_2[D_{C_2}(1)]$, where $i \geq 0$ and $d \in \{0,1\}$. Thus, $c_{i,d}$ defines a nontrivial map

$$c_{i,d} : S^{1+i\rho_{C_2}+d\sigma} \to \underline{\mathbb{F}}_2[D_{C_2}(1)]$$

Along with the unit $\underline{\mathbb{F}}_2[S^0] \to \underline{\mathbb{F}}_2[D_{C_2}(1)]$, these form a basis for $\underline{\mathbb{F}}_2[D_{C_2}(1)]$, and monomials in these generators of length at most k generate $\underline{\mathbb{F}}_2[D_{C_2}(k)]$. Using the commutative diagram

$$\begin{tikzcd} \underline{\mathbb{F}}_2[D_{C_2}(1)^{\wedge 2}] \arrow[r] \arrow[d] & \underline{\mathbb{F}}_2[D_{C_2}(2)] \arrow[d] \\ \underline{\mathbb{F}}_2[(\Sigma M_{C_2}(1))^{\wedge 2}] \arrow[r] & \underline{\mathbb{F}}_2[\Sigma^2 M_{C_2}(2)] \end{tikzcd}$$

we can compute that $c_{0,0} \cdot c_{0,0} \in \underline{\mathbb{F}}_2[D_{C_2}(2)]$ maps to $\Sigma^2(\epsilon_2(1 \otimes 1)) = 0$ in $\underline{\mathbb{F}}_2[\Sigma^2 M_{C_2}(2)]$. Therefore, $c_{0,0} \cdot c_{0,0}$ comes from an element $S^2 \to \underline{\mathbb{F}}_2[D_{C_2}(1)]$. There are no cells $c_{i,d}$ in $\underline{\mathbb{F}}_2[D_{C_2}(1)]$ of degree 2. But, there are three maps $S^2 \to \underline{\mathbb{F}}_2[D_{C_2}(1)]$ coming from the compositions

$$\begin{tikzcd} & \underline{\mathbb{F}}_2[S^{1+\sigma}] \arrow[dr, "c_{0,1}"] & \\ S^2 \arrow[ur, "u_\sigma"] \arrow[r, "a_\sigma"] \arrow[dr, "a_\sigma^2"'] & \underline{\mathbb{F}}_2[S^{2+\sigma}] \arrow[r, "c_{1,0}"] & \underline{\mathbb{F}}_2[D_{C_2}(1)] \\ & \underline{\mathbb{F}}_2[S^{2+2\sigma}] \arrow[ur, "c_{1,1}"'] & \end{tikzcd}$$

i.e. corresponding to the elements $c_{0,1}u_\sigma, c_{1,0}a_\sigma, c_{1,1}a_\sigma^2$. Therefore, $c_{0,0}^2$ is a (possibly zero) linear combination of these three elements (unlike in the nonequivariant case where there was only one candidate nonzero element).[5] In order to determine which linear combination it is, we need some further information which is missing from our analysis thus far.

6. Open Question: Thom spectra and Power Series

Recall that, in the nonequivariant case, we could identify the image of $\mathrm{Sp}_2^2 \wedge H\mathbb{F}_2 \hookrightarrow H\mathbb{F}_2 \wedge H\mathbb{F}_2$ by using the fact that Sp_2^2 is a Thom spectrum on $\mathbf{RP}^\infty$, i.e. $\mathrm{Sp}_2^2 \simeq (B\mathbb{Z}/2)^{1-L}$.

QUESTION 6.1. *Can $D_{C_2}(1)$ be expressed as a Thom spectrum on $B_{C_2}\mathbb{Z}/2$? Is the first attaching map $M_{C_2}(1) \to S^0$ a 'transfer'?*

In fact, the map constructed in [**31**] Prop 4.4 (see section 2.5) is essentially C_2-equivariant (where C_2 acts by complex conjugation), and can be easily used to show

PROPOSITION 6.2. *Let λ denote the one-dimensional $\mathbb{R}[C_2]$-bundle over $B_{C_2}\mathbb{Z}/2$ where $\mathbb{Z}/2$ acts by multiplication by -1. Then there is a map*

$$f : (B_{C_2}\mathbb{Z}/2)^{1-\lambda} \to D_{C_2}(1)$$

which is nonzero on $(H\underline{\mathbb{F}}_2)_0$.

QUESTION 6.3. *Is this map an equivalence after quotienting out the 'lowest cell'? Does this identify $D_{C_2}(1)$ as a Thom spectrum $(B_{C_2}\mathbb{Z}/2)^{1-L}$ where L arises from the nontrivial representation $\mathbb{Z}/2 \to \{\pm 1\}$, and if so, is the action of C_2 on L trivial, or multiplication by -1?*

Even if this question is answered, I am still not sure of how to properly adapt the techniques of section 2.10 to detect elements of $H\underline{\mathbb{F}}_2 \wedge H\underline{\mathbb{F}}_2$. Thanks to the

[5] I believe $c_{0,0}$ corresponds to the element τ_0 of [**19**], in which case this computation is done explicitly there.

decomposition

$$\underline{\mathbb{F}}_2[B_{C_2}\mathbb{Z}/2] \simeq \underline{\mathbb{F}}_2[x, T]/(T^2)$$

where $|T| = \sigma$ and $|x| = \rho_{C_2}$, it seems that instead of dealing with power series in a single variable t of degree 1, we will be dealing with power series in two variables x, T dual to the elements above.

QUESTION 6.4. *Appropriately generalize the methods of section 2.10 to the equivariant setting with maps* $B_{C_2}\mathbb{Z}/2_+ \to H\underline{\mathbb{F}}_2$.

QUESTION 6.5. *Is there a way analogous to (Chapter 2, 10.3) to define the duals of the* equivariant Milnor operations*?*

The elements ξ_i are *indecomposables*, i.e. they generate I/I^2 as a module where I is the kernel in the category of $H\mathbb{F}_2$-modules of the product $H\mathbb{F}_2 \wedge H\mathbb{F}_2 \to H\mathbb{F}_2$. Are the equivariant analogues of these indecomposables as well?

Our current decomposition of $H\underline{\mathbb{F}}_2 \wedge H\underline{\mathbb{F}}_2$ does not recover any *coalgebra* structure. We hope that if the above questions have affirmative answers, then they will provide an alternative explanation.

CHAPTER 6

Further Directions

There are still many answered questions regarding the structure of the equivariant Steenrod algebra, as well as questions regarding the symmetric power filtration itself. In this chapter, we discuss some further directions and questions, some of which are highly conjectural.

1. Generalization to p-power cyclic groups

We have focused on the case when G is a cyclic group of prime order, in particular, $G = C_2$, but one might wonder if there is an analogous decomposition for $H\underline{\mathbb{F}}_2 \wedge H\underline{\mathbb{F}}_2$ when $G = C_4, C_8, C_{16}, \ldots$. Though we have written Chapter 4 for the case where $G = C_p$, we believe we can inductively lift the results to $G = C_{p^2}$, then $G = C_{p^3}$, and so on.

Conjecture 1.1. *When $G = C_{p^n}$, there is a p-local equivalence*

$$\epsilon_k B_G(\mathbb{Z}/p)^k \xrightarrow{\simeq} M_G(k)$$

Proof. (Tentative sketch) Here is an outline for how we believe this proof should go. The first step is to generalize section 4.3, i.e. understanding the structure of

$$\mathrm{Pr}_{C_{p^2}}^{C_{p^2}/1}(S^{\ell\rho_{C_{p^2}}}) := (S^{\ell\rho_{C_{p^2}}}/(S^{\ell\rho_{C_{p^2}}})^{C_p})/C_{p^2}$$

That is, take the one-point compactification of a direct sum of ℓ copies of the regular representation, quotient out the points with nontrivial isotropy (i.e. the points fixed by $C_p \subset C_{p^2}$), and then take the orbit space under the free C_{p^2} action. This single computation would allow us to generalize the cofiber sequences of sections 4.4 and

4.5 up to $G = C_{p^2}$. Section 4.6, and in particular, the computation of the G-fixed points of $\epsilon_k B_G(\mathbb{Z}/p)^k$, should lift to the C_{p^2}-equivariant setting quite easily - we hope that the resulting expression lines up with the decomposition we'd get from section 4.5. Lastly, we would need to generalize the argument from section 4.8, explicitly proving that the first cofiber satisfies $M_{C_{p^2}}(1) \simeq \epsilon_1 B_{C_{p^2}}(\mathbb{Z}/p)$.

Each successive lift, from C_p to C_{p^2}, then from C_{p^2} to C_{p^3}, requires only that we show an equivalence on the level of geometric fixed points - hence, the inductive approach. $\square$

The other main ingredient is a generalization of Chapter 5. The first step is as follows.

QUESTION 1.2. *Is there a decomposition*

$$B_{C_{2^n}}(\mathbb{Z}/2) \wedge H\underline{\mathbb{F}}_2 \simeq \bigvee_V S^V \wedge H\underline{\mathbb{F}}_2$$

where V ranges over representations of C_{2^n}?

To answer this question, we would need to generalize the result of section 3.4, i.e. obtain an explicit simplicial model for $E_G\Lambda$ when $G = C_{p^n}$ and $\Lambda = \Delta_k$.

2. The Reduction Theorem

One motivation for lifting these results to the case of $G = C_{2^n}$ is the so-called 'Reduction theorem' of [**18**].First, recall that MU denotes the *complex cobordism spectrum.* Its homotopy groups form the ring

$$\pi_*\mathrm{MU} = \mathbb{Z}[c_1, c_2, c_3, \ldots]$$

where $|c_i| = 2i$. When we p-localize, MU splits into a wedge sum of suspensions of BP, the *Brown-Peterson spectrum* whose homotopy groups are

$$\pi_*\mathrm{BP} = \mathbb{Z}_{(p)}[v_1, v_2, v_3, \ldots]$$

where $|v_j| = 2(p^j - 1)$ (i.e., $v_j = c_{p^j-1}$). There is an explicit C_{2^n}-equivariant lift $\mathrm{MU}^{((C_{2^n}))}$ - for example, when $n = 1$, we get the C_2-spectrum $\mathrm{MU}_{\mathbb{R}}$, and when we 2-localize, it splits similarly as a wedge sum of suspensions of $\mathrm{BP}_{\mathbb{R}}$. The Reduction theorem is a natural generalization to this setting of Quillen's computation of the homotopy groups of MU.

THEOREM 2.1. *([**18**], theorem 6.5) Let $G = C_{2^n}$. The map $\mathrm{MU}^{((G))} \to H\underline{\mathbb{Z}}_2$ lifts to the dotted map*

$$\begin{tikzcd} \mathrm{MU}^{((G))} \arrow[r] \arrow[d] & H\underline{\mathbb{Z}}_2 \\ \mathrm{MU}^{((G))} \wedge_A S^0 \arrow[ur, dashed] & \end{tikzcd}$$

where A is the twisted monoid ring generated by the elements $\overline{r}_1, \overline{r}_2, \ldots$. Then this map is an equivalence.[1]

Let us first discuss an application of this theorem. In the case $G = C_2$, one can use this theorem to calculate $H\underline{\mathbb{F}}_2 \wedge_{\mathrm{BP}_{\mathbb{R}}} H\underline{\mathbb{F}}_2$. The argument is roughly as follows: consider the cofiber sequence

$$\Sigma^{|\overline{v}_j|}\mathrm{BP}_{\mathbb{R}} \xrightarrow{\overline{v}_j} \mathrm{BP}_{\mathbb{R}} \longrightarrow \mathrm{BP}_{\mathbb{R}}/(\overline{v}_j)$$

After we apply $- \wedge_{\mathrm{BP}_{\mathbb{R}}} H\underline{\mathbb{F}}_2$ to this sequence, one can show the first map becomes zero, and thus from the long exact sequence one can conclude

$$(\mathrm{BP}_{\mathbb{R}}/(\overline{v}_j)) \wedge_{\mathrm{BP}_{\mathbb{R}}} H\underline{\mathbb{F}}_2 \simeq H\underline{\mathbb{F}}_2 \vee \Sigma^{|\overline{v}_j|+1} H\underline{\mathbb{F}}_2$$

i.e., we get a new element of degree $|\overline{v}_j| + 1 = (2^j - 1)\rho_{C_2} + 1$. We call this element $\overline{\tau}_j$. In this way, by 'reducing' from $\mathrm{BP}_{\mathbb{R}}$ all the way down to $H\underline{\mathbb{F}}_2$ by killing the $\overline{v}_j$'s, we obtain the elements $\overline{\tau}_j$. The differential map in each of the cofiber sequences

[1] $\overline{r}_i$ is of degree $i\rho_{C_2}$, and is just an equivariant lift of the classical element $c_i \in \mathrm{MU}_{2i}$. When $i = 2^j - 1$, we get the element $\overline{v}_j$ which is just an equivariant lift of the element $v_j \in \mathrm{BP}_{2(2^j-1)}$.

above corresponds to the dual of $\overline{\tau}_j$, i.e. a *equivariant Milnor operation* $H\underline{\mathbb{F}}_2 \to \Sigma^{(2^j-1)\rho_{C_2}+1} H\underline{\mathbb{F}}_2$.

Numerical evidence from the computation (Chapter 5, 4.1) suggests that in the symmetric power filtration, the elements $\overline{\xi}_j, \overline{\tau}_{j-1}$ first appear at the j-th stage of the filtration, i.e. $D_{C_2}(j)$. Thus we wonder,

QUESTION 2.2. *Is there a correspondence between the successive stages of the symmetric power filtration and the successive stages of the map coming from the Reduction Theorem? If we lift our results to C_{2^n}, will this correspondence still hold?*

In the proof of the above theorem, Hill, Hopkins, and Ravenel focus on the geometric fixed points of the dotted map, proceeding by induction on n (the $n = 0$ case is a classical result of Quillen). For example, when $n = 1$, $G = C_2$, and the key technical point is arguing that the reduction map which kills all of the remaining $\overline{r}_i$'s,

$$\Phi^G(\mathrm{MU}_{\mathbb{R}}/(\overline{r}_{2^k-1})) \to \Phi^G H\underline{\mathbb{Z}}_{(2)}$$

is an isomorphism on π_{2^k}. $\pi_*\Phi^G H\underline{\mathbb{Z}}_{(2)} \simeq \mathbb{Z}/2[b]$ where $b = u_{2\sigma}a_\sigma^{-2}$, so they are trying to show that an element on the homotopy of the left side above hits $b^{2^{k-1}}$ on the right. The proof is computational.

However, we've given an alternative interpretation of this element $b^{2^{k-1}}$ in (Chapter 4, 4.1): it comes from the 2^k-cell of $H\mathbb{Z} \wedge \Sigma B\mathbb{Z}/2_+$ (the right side of the cofiber sequence), which arises from the $2^k - 1$-cell of the copy of $B\mathbb{Z}/2_+$. If there is indeed a correspondence between the stages of the equivariant symmetric power filtration and the stages of the reduction map, then

QUESTION 2.3. *(Highly conjectural) Can we find a more conceptual proof of the Reduction theorem using our computation in (Chapter 4, 4.1)?*

One motivation for answering these two questions, is that the symmetric power approach works for *odd primes*, whereas we don't have a good description for the C_p-equivariant analogues of MU and BP!

3. Odd primes

Another obvious next step is finding an analogous decomposition of $H\underline{\mathbb{F}}_p \wedge H\underline{\mathbb{F}}_p$ at odd primes. The primary hurdle here is a generalization of section 5.3, namely, computing a decomposition of $B_{C_p}(\mathbb{Z}/p) \wedge H\underline{\mathbb{F}}_p$. Explicit computation shows that the summands which appear do NOT appear to be of the form

$$\mathbb{F}_p[C_p] \longrightarrow \cdots \longrightarrow \mathbb{F}_p[C_p] \longrightarrow \mathbb{F}_p$$

In particular, one difference from the $p = 2$ case is that there are no representation spheres of C_p of dimension 1 other than the trivial representation sphere S^1, because the other irreducible real representations of C_p are *two-dimensional*. It still remains a mystery as to how one should describe these summands. Some current work by D. Wilson (the author of [**44**]) may shed light in this direction.

4. The Whitehead Conjecture

The *Whitehead conjecture* is the following incredible property of the symmetric power filtration. It was originally proven in [**21**] and [**23**], and more recently proven in a more unifying fashion in [**22**].

THEOREM 4.1. *The following two statements are equivalent and true.*

- *Each map* $\mathrm{Sp}^{p^{k-1}} \to \mathrm{Sp}^{p^k}$ *is zero on p-local homotopy groups in positive degrees.*
- *The sequence* $\cdots \to L(2) \to L(1) \to S^0 \to H\mathbb{Z}$ *is exact on homotopy groups.*

The second bullet point of the Whitehead conjecture says that the $\Omega^\infty L(k)$'s give a resolution of $\mathbb{Z}$ by modules over the *Dyer-Lashof algebra*. Also discussed in [**22**] is the fact that the exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \Omega^\infty \Sigma L(2) \arrow[r] \arrow[l, dashed, bend right] & \Omega^\infty \Sigma L(1) \arrow[r] \arrow[l, dashed, bend right] & S^1 \arrow[l, dashed, bend right] \end{tikzcd}$$

splits, although the splitting maps are not infinite loop maps. These maps actually arise from the Goodwillie tower of S^1. There is therefore a *duality* between the symmetric power filtration and the Goodwillie tower for a sphere.

QUESTION 4.2. *(Highly conjectural) Is there an equivariant analogue of this duality?*

Goodwillie calculus in the equivariant setting provides us not a tower, but a *tree* indexed on finite G-sets ([**?**]). Therefore, if were to be a candidate for this duality, it should be indexed on finite G-sets.

The first bullet point of the Whitehead conjecture implies that the map $S^0 \to H\mathbb{Z}$ is an equivalence on π_0 and kills all higher homotopy groups. In the equivariant case, the Mackey functor homotopy group $\underline{\pi}_0(S^0)$ is isomorphic to $\underline{\mathcal{A}}$, the *Burnside ring Mackey functor*. So if we are searching for an equivariant version of the Whitehead conjecture, it may be that we wish to build a filtration for $H\underline{\mathcal{A}}$. The following is an idea. Let Fin be the category of finite sets with injections. If $\mathcal{C}$ is a sufficiently nice category (for example, spaces, spectra, G-spaces, or G-spectra), then we can construct $\mathrm{Sp}^\infty X$ for any $X \in \mathcal{C}$ by taking the colimit of the diagram

$$X^\bullet : \mathrm{Fin} \to \mathcal{C}$$

What happens if we instead consider the functor $X^\bullet : \mathrm{Fin}_G \to \mathcal{C}$, where Fin_G is the category of finite *G-sets* with injections?

QUESTION 4.3. *If we perform the above construction with $X = \mathbb{S}_G$, do we get the Eilenberg-Maclane spectrum of the Burnside Mackey functor?*

Certainly the object we get is filtered by subobjects corresponding to the finite G-sets. So perhaps this is a candidate for an analogous duality.

5. Other Mackey functors

We used the symmetric power filtration to analyze the structure of $H\underline{\mathbb{Z}}$ and $H\underline{\mathbb{F}}_p$. One may ask if there is a similar filtration for $H\underline{M}$ for $\underline{M}$ a *different* Mackey (or Tambara) functor. There are multiple fundamental building blocks in the category of p-local Mackey functors - thus there are different notions of the equivariant dual Steenrod algebra. As suggested above, we have a putative construction for $H\underline{\mathcal{A}}$, where $\mathcal{A}$ is the Burnside ring Mackey functor, with the ring structure arising out of the operations $\sqcup$ and $\times$ on Fin_G. Perhaps this filtration can be analyzed via its cofibers as well.

Question 5.1. *Consider the colimit of the diagram*

$$\mathbb{S}_G^{\times\bullet} : \mathrm{Fin}_G \to \mathscr{S}^G$$

It has a natural subobject corresponding to any finite G-set T by restricting the diagram to those G-sets mapping into T. Thus, we have a filtration indexed on the finite G-sets. Calculate the cofibers. Do they split after smashing with $H\underline{\mathcal{A}}$?

Bibliography

[1] G. Arone, *A Branching Rule for Partition Complexes*

[2] G. Arone, W. Dwyer, *Partition Complexes, Tits Buildings and Symmetric Products*, Proc. London Math. Soc. (3) **82** (2001), no. 1, pp. 229-256

[3] G. Arone, W. Dwyer, K. Lesh, *Bredon Homology of Partition Complexes*, (2013) arXiv:1306.0056

[4] G. Arone, M. Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. **135** (1999), No. 3, pp. 743-788

[5] D. Benson, *Representations and Cohomology, II*

[6] M. Behrens, *The Goodwillie tower and the EHP sequence*, Mem. Amer. Math. Soc. **218** (2012), No. 1026

[7] L. Brantner, *Parabolic Restrictions of Bruhat-Tits Buildings*, to appear

[8] H. Cartan, *Sur l'itÃľration des opÃľrations de Steenrod*, Commentarii Mathematici Helvetici, **29** No. 1 (1955), 40-58

[9] J. Caruso, *Operations in equivariant $\mathbb{Z}/p$-cohomology*, Math. Proc. Cambridge Phil. Soc. **126** (1999), No. 3, pp. 521-541

[10] S. R. Costenoble and S. Waner, *The Equivariant Thom Isomorphism*, Pacific J. Math. **152** (1992), 21-39

[11] C. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience Publishers, New York (1962)

[12] A. Dold, R. Thom, *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math. 67 (1958), pp. 239-281

[13] P. Dos Santos, *A Note on the Equivariant Dold-Thom Theorem*, J. of Pure and Appl. Alg. **183** (2003), No. 1-3, pp. 299-312

[14] S. Eilenberg, N. E. Steenrod, *Axiomatic approach to homology theory*, Proc. Natl. Acad. Sci. **31** (1945), pp. 117-120

[15] A. Elmendorf, I. Kriz, M. Mandell, J. P. May, with an appendix by M. Cole, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, no. 47, AMS. Providence, RI, 1997

[16] A. Hatcher, *Algebraic Topology*

[17] M. Hausmann, *Symmetric Products and Subgroup Lattices* (2016), arXiv:1607.07559

[18] M. Hill, M. Hopkins, D. Ravenel, *On the Non-existence of Elements of Kervaire Invariant One*, Annals of Mathematics **184** (2016), No. 1, pp. 1-262

[19] P. Hu, I. Kriz, *Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence*, Topology **40** (2001), 317-399

[20] D. S. Kahn, S. B. Priddy, *The transfer and stable homotopy theory*, Math. Proc. Cambridge Philos. Soc **83** (1978), pp. 103-111

[21] N. Kuhn, *A Kahn-Priddy sequence and a conjecture of G. W. Whitehead*, Math. Proc. Cambridge Phil. Soc. **92** (1982), No. 3, pp. 467-483

[22] N. Kuhn, *The Whitehead conjecture, the Tower of S^1 conjecture, and Hecke Algebras of type A*, Journal of Topology **8** (2015), Issue 1, pp. 118-146

[23] N. Kuhn, S. Priddy, *The transfer and Whitehead's conjecture*, Math. Proc. Cambridge Phil. Soc. **98** (1985), Issue 3, pp. 459-480

[24] N. Kuhn, S. Mitchell, S. Priddy, *The Whitehead Conjecture and Splitting $B(\mathbb{Z}/2)^k$*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), No. 1, 255-258

[25] J. Lurie, Notes from MIT's 18.917 course *Topics in Algebraic Topology: The Sullivan Conjecture* (Fall 2007), available at https://ocw.mit.edu/courses/mathematics/18-917-topics-in-algebraic-topology-the-sullivan-conjecture-fall-2007/lecture-notes/

[26] M. Mandell, J. P, May, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc. **159** (2002), No. 755

[27] J. P. May, *Equivariant Homotopy and Cohomology theory*, CBMS Regional Conference Series in Mathematics **91** (1996), 366 pp.

[28] J. P. C. Greenlees, J. P, May, *Equivariant Stable Homotopy theory*, in Ioan James (ed.), *Handbook of Algebraic Topology*, pp. 279-325 (1995)

[29] H. Miller, *An Algebraic Analogue of a Conjecture of G. W. Whitehead*, Proc. Amer. Math. Soc. **84** (1982), No. 1

[30] J. Milnor, *The Steenrod Algebra and its Dual*, Annals of Mathematics, Second Series **87** (1957), 150-171

[31] S. Mitchell, S. Priddy, *Stable Splittings Derived from the Steinberg Module*, Topology **22** (1983), Issue 3, 285-298

[32] M. Nakaoka, *Cohomology mod p of symmetric products of spheres*, J. Inst. Poltech. Osaka City Univ. Ser. A **9** (1958), No. 1, 1-18

[33] D. Quillen, *Rational Homotopy Theory*, Annals of Mathematics, **90** (1969), No. 2, 205-295

[34] S. Schwede, *Equivariant properties of Symmetric Products*, arXiv:1403.1355

[35] S. Schwede, *Lectures on Equivariant Stable Homotopy Theory*

[36] S. Schwede, B. Shipley, *Stable Model Categories are Categories of Modules*, Topology **42** (2003), 103-153

[37] N. Steenrod, *Products of cocycles and extensions of mappings*, Annals of Mathematics, Second Series, **48** (1947), 290-320

[38] N. Steenrod, *Homology groups of symmetric groups and reduced power operations*, Proc. National Academy of Sciences, **39** (1953), 213-217

[39] R. Steinberg, *Prime power representations of finite linear groups II*, Can. J. Math. **18** (1956), pp. 580-591

[40] N. Strickland, *Tambara Functors*

[41] D. Tambara, *On multiplicative transfer*, Comm. Algebra **21** (1993), No. 4, 1393-1420

[42] S. Takayasu, *On Stable summands of Thom spectra of* $B(\mathbb{Z}/2)^n$ *associated to Steinberg modules*, J. Math. Kyoto Univ. **39** (1999), No. 2, 377-398

[43] J. Ullman, *Tambara Functors and Commutative Ring Spectra*

[44] D. Wilson, *Power operations for* $\mathrm{H}\underline{\mathbb{F}}_2$ *and a cellular construction of* $\mathrm{BP}_{\mathbb{R}}$, in progress