MATH 220.201 CLASS 9 QUESTIONS

1. Prove that the equation $x^6 + x^4 + 2x^2 + 1 = 0$ has no real solutions.

   Direct Proof. We wish to show that $\nexists x \in \mathbb{R}, x^6 + x^4 + 2x^2 + 1 = 0$. We will prove the equivalent statement that $\forall x \in \mathbb{R}, x^6 + x^4 + 2x^2 + 1 \neq 0$.

   For any real number x,

   $$x^6 + x^4 + 2x^2 + 1 = (x^3)^2 + (x^2)^2 + 2x^2 + 1 \geq 0 + 0 + 2(0) + 1 = 1$$

   Therefore, $x^6 + x^4 + 2x^2 + 1 \geq 1$. So $x^6 + x^4 + 2x^2 + 1 \neq 0$, as desired. □

   Proof by Contradiction. Suppose for a contradiction that there exists a real number x such that $x^6 + x^4 + 2x^2 + 1 = 0$. Then

   $$-1 = x^6 + x^4 + 2x^2 = (x^3)^2 + (x^2)^2 + 2x^2 \geq 0 + 0 + 2(0) = 0$$

   because x is a real number. So $-1 \geq 0$, which is a contradiction! Therefore, there is no such x. □

2. Let x be a nonzero real number. If $x + \frac{1}{x} < 2$, then $x < 0$.[1]

   Note that in symbols, the statement is

   $$\forall x \in \mathbb{R} \text{ s.t. } x \neq 0, ((x + \frac{1}{x} < 2) \implies (x < 0))$$

   Proof by Contradiction. Suppose that there exists $x \neq 0$ such that $x + \frac{1}{x} < 2$ and $x \geq 0$. Then

   $$\begin{aligned} x + \frac{1}{x} < 2 &\implies x^2 + 1 < 2x \qquad \text{because } x \geq 0 \\ &\implies x^2 - 2x + 1 < 0 \\ &\implies (x-1)^2 < 0 \end{aligned}$$

   But $(x-1)^2 \geq 0$, because $x - 1 \in \mathbb{R}$. Therefore we have a contradiction! □

[1] You can adapt your argument to prove the following well-known theorem.

Arithmetic Mean - Geometric Mean Inequality: For any positive real numbers x and y, $\sqrt{xy} \leq \frac{x+y}{2}$.

Proof. Since $\sqrt{x} - \sqrt{y}$ is a real number,

$$(\sqrt{x} - \sqrt{y})^2 \geq 0 \implies x - 2\sqrt{xy} + y \geq 0 \implies x + y \geq 2\sqrt{xy} \implies \frac{x+y}{2} \geq \sqrt{xy}$$

□

Proof by Contrapositive. We show that if $x \geq 0$, then $x + \frac{1}{x} \geq 2$. Suppose that $x \geq 0$. First, since x is assumed to be nonzero, it follows that $x > 0$. Then

$$\begin{aligned} (x-1)^2 \geq 0 &\implies x^2 - 2x + 1 \geq 0 \\ &\implies x^2 + 1 \geq 2x \\ &\implies x + \frac{1}{x} \geq 2 \qquad \text{because } x > 0 \end{aligned}$$

which completes the proof. □

3. Let x be an irrational number. Then there is no **largest** rational number y with the following property: $y \leq x$.

In symbols, the statement is saying that

$$\forall x \in \mathbb{R} - \mathbb{Q}, \nexists y \in \mathbb{Q}, (y \leq x \wedge \forall z \in \mathbb{Q}, (z \leq x \implies z \leq y))$$

Proof by Contradiction. Suppose for a contradiction that there is such an irrational number x and a rational number y such that $y \leq x$ and y is the largest among all rational numbers with this property. Since x is irrational and y is rational, they are unequal, so $y < x$. Thus, $x - y > 0$ Then there is some positive integer n such that $x - y > \frac{1}{n}$.[2] Then $x > y + \frac{1}{n}$, and $y + \frac{1}{n}$ is rational. So by the assumption, $y + \frac{1}{n} \leq y$. Thus, $\frac{1}{n} \leq 0$. This is clearly false, so we have a contradiction! □

Direct Proof. It suffices to show that for any irrational x and for any rational y such that $y \leq x$, there exists some rational z such that $y < z \leq x$. That is, we are proving the equivalent statement

$$\forall x \in \mathbb{R} - \mathbb{Q}, \forall y \in \mathbb{Q}, (y \leq x \implies \exists z \in \mathbb{Q}, (z \leq x \wedge z > y))$$

Suppose that y is a rational number such that $y \leq x$. Since x is irrational and y is irrational, they are unequal, and so $y < x$. Thus, $x - y > 0$. Then, there is some positive integer n such that $x - y > \frac{1}{n}$. Now take $z = y + \frac{1}{n}$. Clearly $z > y$, z is rational (because it is the sum of two rational numbers), and it is also true that $x > z$ because $x - y > \frac{1}{n}$. Thus, we have constructed a valid z, as desired. □

Use the following theorem in questions 4 and 5:

Intermediate Value Theorem: For every continuous function f on the closed interval $[a, b]$, and for every number k between $f(a)$ and $f(b)$, there is some $c \in [a, b]$ such that $f(c) = k$.

[2] The proof of this is delving into the axiomatic construction of the real numbers, as limits of *Cauchy sequences* of rational numbers. For now I will justify it as follows: if $x - y \leq \frac{1}{n}$ for *every* natural number n, then this would mean that $x - y \leq \lim_{n \to \infty} \frac{1}{n} = 0$. However, we've assumed that $x - y > 0$, so this is a contradiction.

4.
- The equation $x^5 + 2x - 5 = 0$ has a solution on the interval $[1, 2]$.

  Proof. Let $f(x) = x^5 + 2x - 5$. The Intermediate Value Theorem tells us that

  For every number k between $f(1)$ and $f(2)$, there is some $c \in [1, 2]$ such that $f(c) = k$.

  $f(1) = 1^5 + 2(1) - 5 = -3$ and $f(2) = 2^5 + 2(2) - 5 = 31$. The Intermediate Value Theorem therefore implies

  For every number k between -3 and 31, there is some $c \in [1, 2]$ such that $f(c) = k$.

  Since 0 is between -3 and 31, it follows that there is some $c \in [1, 2]$ such that $f(c) = 0$. This is the desired statement. □

- The equation $x^5 - 2x - 5 = 0$ has *exactly one* solution on the interval $[1, 2]$.

Proof. Suppose there are two distinct real numbers $c, d \in [1, 2]$ such that $c^5 + 2c - 5 = d^5 + 2d - 5 = 0$. Then $c^5 + 2c = d^5 + 2c$. Either $c > d$ or $d > c$, so without loss of generality, $c > d$. Then $c^5 > d^5$ and $2c > 2d$. Thus, $c^5 + 2c - 5 > d^5 + 2d - 5$. We have a contradiction! □

5. Any polynomial equation $f(x) = 0$ of odd degree has a real number solution. (Note: this proof is much more involved than anything you'll be expected to prove at this point in the course.)

Proof. We begin by proving a lemma.

Lemma 0.1. *Let $f(x)$ be an odd-degree polynomial. Then there is some real number a such that $f(a) > 0$, and some real number b such that $f(b) < 0$.*

Proof of Lemma. Since f has odd degree, its degree is equal to $2k + 1$ for some nonnegative integer k. Then, by the definition of a polynomial, we can write

$$f(x) = a_{2k+1}x^{2k+1} + a_{2k}x^{2k} + \ldots + a_1 x + a_0 = \sum_{i=0}^{2k+1} a_i x^i$$

for some real numbers $a_0, a_1, \ldots, a_{2k+1}$ such that $a_{2k+1} \neq 0$. If the required conclusion holds for a polynomial $f(x)$, then it holds for the polynomial $t \cdot f(x)$ for any nonzero real number t. Therefore, we may assume without loss of generality that $a_{2k+1} = 1$. Now consider plugging in $x = 1 + |a_{2k}| + |a_{2k-1}| + \ldots + |a_1| + |a_0|$.

Then

$$
\begin{aligned}
x = 1 + |a_{2k}| + \ldots + |a_0| &\geq 1 + |a_{2k}| + |\frac{a_{2k-1}}{x}| + \ldots + |\frac{a_0}{x^{2k}}| && \text{because } x \geq 1 \\
&> |a_{2k}| + |\frac{a_{2k-1}}{x}| + \ldots + |\frac{a_0}{x^{2k}}| \\
x^{2k+1} &> |a_{2k}x^{2k}| + |a_{2k-1}x^{2k-1}| + \ldots + |a_0| && \text{because } x \text{ is positive} \\
x^{2k+1} &> |a_{2k}x^{2k} + a_{2k-1}x^{2k-1} + \ldots + a_0| && \text{(Triangle inequality)}
\end{aligned}
$$

Thus, it follows that $x^{2k+1} + a_{2k}x^{2k} + a_{2k-1}x^{2k-1} + \ldots + a_0 > 0$. This is our value '$a$'. To get our value '$b$', take $x = -1 - |a_{2k}| - \ldots - |a_0|$, and use similar reasoning. □

By the lemma, there exist real numbers a, b such that $f(a) > 0$ and $f(b) < 0$. 0 is therefore between $f(a)$ and $f(b)$, so by the Intermediate Value Theorem, there is some real number c in the interval between a and b such that $f(c) = 0$. This completes the proof. □