MATH 220.201 CLASS 5 SOLUTIONS

1. Let $n \in \mathbb{Z}$. Prove that if n is odd, then $n^2 - 5n + 2$ is even.

Proof. If n is an odd integer, then there is some integer k such that $n = 2k + 1$. Then

$$\begin{aligned} n^2 - 5n + 2 &= (2k+1)^2 - 5(2k+1) + 2 \\ &= 4k^2 + 4k + 1 - 10k - 5 + 2 \\ &= 4k^2 - 6k - 2 \\ &= 2(2k^2 - 3k - 1) \end{aligned}$$

Since k is an integer, $2k^2 - 3k - 1$ is an integer. Therefore, $n^2 - 5n + 2$ is an integer. □

2. Let $n \in \mathbb{Z}$. Prove that n is odd if and only if n^2 is odd.

Proof. Let n be an integer. We first prove that if n is odd, then n^2 is odd. If n is odd, then $n = 2k + 1$ for some integer k. Then

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Since k is an integer, $2k^2 + 2k$ is an integer. Therefore, n^2 is odd.

Now, we prove that if n^2 is odd, then n is odd. We do so by proving the contrapositive: namely showing that if n is even, then n^2 is even.[1] If n is even, then $n = 2\ell$ for some integer ℓ. Then

$$n^2 = (2\ell)^2 = 4\ell^2 = 2(2\ell^2)$$

Since ℓ is an integer, $2\ell^2$ is an integer. Therefore, n^2 is even. □

3. Let $n \in \mathbb{Z}$. Prove that if $7n + 4$ is even, then $3n - 11$ is odd.

Proof. We will show that[2]

$$(7n + 4 \text{ is even} \implies n \text{ is even}) \text{ and } (n \text{ is even} \implies 3n - 11 \text{ is odd})$$

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[1] Remember, we assumed n is an integer.

[2] A logical maneuver we are using here is ***transitivity***, namely that for any statements (or open sentences) P, Q, R,

$$((P \implies Q) \wedge (Q \implies R)) \implies (P \implies R)$$

If you want a fun exercise, you can either express the above statement entirely in terms of $\wedge, \vee, \sim$ and show it's a tautology, or write a truth table for it.

and deduce the conclusion.

Let's first prove the one on the left.[3] We show it by proving the contrapositive. Namely, we show that if n is an odd integer, then $7n+4$ is odd. If n is odd, then $n = 2k+1$ for some integer k. Then

$$7n + 4 = 7(2k + 1) + 4 = 14k + 11 = 2(7k + 5) + 1$$

Since k is an integer, $7k + 5$ is an integer. Therefore, $7n + 4$ is odd, as desired.

Now we show that if n is even, $3n - 11$ is odd. If n is even, then $n = 2\ell$ for some integer ℓ. Then

$$3n - 11 = 3(2\ell) - 11 = 6\ell - 11 = 2(3\ell - 6) + 1$$

Since ℓ is an integer, $3\ell - 6$ is an integer. Therefore, $3n - 11$ is odd, as desired. □

Proof. Here is an alternate proof. Suppose that $7n-4$ is even. Then $7n-4 = 2k$ for some integer k. Therefore

$$\begin{aligned} 3n - 11 = 7n - 4n - 4 - 7 &= (7n - 4) - 4n - 7 \\ = 2k + 2(-2n - 4) + 1 &= 2(k - 2n - 4) + 1 \end{aligned}$$

Since k and n are integers, $k-2n-4$ is an integer. Therefore, $3n-11$ is odd.[4] □

4. Suppose that the following fact is known to be true[5]

Lemma 0.1. *For every $k \in \mathbb{Z}$, $k(k + 1)$ is an even integer.*

Prove that if n is any odd integer, then $n^2 - 1$ is a multiple of 8.

Proof. Suppose that n is an odd integer. Then $n = 2k + 1$ for some integer k. Then

$$n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k(k + 1)$$

k is an integer, so by the lemma, $k(k+1)$ is an even integer. Therefore, $k(k+1) = 2\ell$ for some integer ℓ. Then $n^2 - 1 = 4(2\ell) = 8\ell$. Therefore, $n^2 - 1$ is a multiple of 8. □

[3] Note, we could prove these two in either order.

[4] This is a proof where the way you'd figure it out is the reverse of the finished product. The idea of this proof is that you prove that the difference between $7n - 4$ and $3n - 11$ is *always odd*.

[5] If you are curious how to prove this particular lemma, here is a rigorous proof.

Proof. If k is an integer, then it is either an even integer, or it is an odd integer.

If k is even, then $k = 2a$ for some integer a. Then $k(k + 1) = 2a(2a + 1) = 2(2a^2 + a)$. Since a is an integer, $2a^2 + a$ is an integer, and so $k(k + 1)$ is even.

If k is odd, then $k = 2b + 1$ for some integer b. Then $k(k + 1) = (2b + 1)(2b + 2) = 2(2b^2 + 3b + 1)$. Since b is an integer, $2b^2 + 3b + 1$ is an integer, and so $k(k + 1)$ is even. □

5. Let $n \in \mathbb{Z}$. Prove that $n^2 - 3n + 9$ is odd.

Proof. If n is an integer, then n is even or n is odd. We consider these two cases separately.

Case 1, n is even: If n is even, then $n = 2k$ for some integer k. Then

$$n^2 - 3n + 9 = (2k)^2 - 3(2k) + 9 = 4k^2 - 6k + 9 = 2(2k^2 - 3k + 4) + 1$$

Since k is an integer, $2k^2 - 3k + 4$ is an integer. Therefore, $n^2 - 3n + 9$ is odd.

Case 2, n is odd: If n is odd, then $n = 2k + 1$ for some integer k. Then

$$\begin{aligned} n^2 - 3n + 9 &= (2k + 1)^2 - 3(2k + 1) + 9 \\ &= 4k^2 + 4k + 1 - 6k - 3 + 9 \quad = 2(2k^2 - k + 3) + 1 \end{aligned}$$

Since k is an integer, $2k^2 - k + 3$ is an integer. Therefore, $n^2 - 3n + 9$ is odd. □

Proof. Here is an alternate proof. Notice that

$$\begin{aligned} n^2 - 3n + 9 &= (n^2 - 3n + 2) + 7 = (n - 1)(n - 2) + 7 \\ &= (n - 2)(n - 2 + 1) + 7 \end{aligned}$$

By Lemma , $(n - 2)(n - 2 + 1)$ is even. Therefore, it can be written in the form 2ℓ for some integer ℓ. Thus, $n^2 - 3n + 9 = 2\ell + 7 = 2(\ell + 3) + 1$. Since ℓ is an integer, $\ell + 3$ is an integer, and therefore $n^2 - 3n + 9$ is odd. □

6. Let $a, b \in \mathbb{Z}$. Prove that

$$ab \text{ is even} \iff (a \text{ is even}) \vee (b \text{ is even})$$

Proof. $\Rightarrow$: We first prove the forwards direction by proving its contrapositive[6]

$$(a \text{ is odd}) \wedge (b \text{ is odd}) \implies ab \text{ is odd}$$

If a and b are odd, then $a = 2k + 1$ and $b = 2\ell + 1$ for some integers k and ℓ. Then

$$ab = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell) + 1$$

Since k and ℓ are integers, $2k\ell + k + \ell$ is an integer. Therefore, ab is odd.

$\Leftarrow$: We prove the backwards direction by considering two possible cases.

If a is even, then $a = 2k$ for some integer k. Then $ab = (2k)b = 2(kb)$. Since k and b are integers, kb is an integer. Therefore, ab is even.

If b is even, then $b = 2\ell$ for some integer ℓ. Then $ab = a(2\ell) = 2(a\ell)$. Since a and ℓ are integers, $a\ell$ is an integer. Therefore, ab is even. □

[6] I have used De Morgan's law here.

7. The following is a faulty proof. Explain what is wrong with it.

Proposition 0.2. *If m is an even integer and n an odd integer, then $3m+n+mn-1$ is a multiple of 4.*

Proof. Let m be an even integer and n an odd integer. Then $m = 2k$ and $n = 2k+1$ for some $k \in \mathbb{Z}$. Therefore

$$\begin{aligned} 3m+n+mn-1 &= 3(2k)+(2k+1)+2k(2k+1)-1 \\ &= 6k+2k+1+4k^2+2k-1 \\ &= 4k^2+8k \\ &= 4(k^2+2k) \end{aligned} \tag{0.1}$$

Since k^2+2k is an integer, $3m+n+mn-1$ is a multiple of 4. □

Solution: The issue is that if m is an arbitrary even integer and n is an arbitrary odd integer, then $m = 2k$ and $n = 2\ell+1$ where k and ℓ are not necessarily equal. In the faulty proof above, the same letter was used for both. The proposition is actually false: try $m = 2$ and $n = 3$.

8. Let $a, b, c, d, n \in \mathbb{Z}$. Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $ac \equiv bd \pmod{n}$, where the notation '$\equiv \pmod{n}$' is defined below.

Definition 0.3. *For integers a, b, n, if $a-b$ is a multiple of n, let us write this as*

$$n|(a-b) \text{ ; `}n \text{ divides } a-b\text{', } or$$

$$a \equiv b \pmod{n} \text{ ; `}a \text{ is congruent to } b \text{ modulo } n\text{'.}$$

Proof. If $a \equiv b \pmod{n}$, then $a-b = kn$ for some integer k. Similarly $c-d = \ell n$ for some integer ℓ. Then $a = b+kn$ and $c = d+\ell n$, so

$$\begin{aligned} ac-bd &= (b+kn)(d+\ell n)-bd \\ &= bd+kdn+b\ell n+k\ell n^2-bd \\ &= (kd+b\ell+k\ell n)n \end{aligned}$$

Since k, ℓ, n are integers, $kd+b\ell+k\ell n$ is an integer. Therefore, $ac \equiv bd \pmod{n}$. □