MATH 220.201 CLASS 4 QUESTIONS

(1) Show that $\sim(P \iff Q) \equiv (P \wedge \sim Q) \vee (Q \wedge \sim P)$.

Solution: You can do this by a truth table in P and Q. Or, you can use the rules of logic to prove it, as follows (the middle step is De Morgan's law):

$$\begin{aligned} \sim(P \iff Q) &\equiv \sim((P \implies Q) \wedge (Q \implies P)) \\ &\equiv \sim(P \implies Q) \vee \sim(Q \implies P) \\ &\equiv (P \wedge \sim Q) \vee (Q \wedge \sim P) \end{aligned}$$

(2) Let $P(n)$ be the open sentence

$$P(n) : \frac{5n-3}{6} \text{ is an integer}$$

over the domain $n \in \mathbb{Z}$.

(a) Is the following true or false? $\exists n \in \mathbb{Z}, P(n)$. Explain.

Solution: True, because $P(3)$ is true. (i.e. $\frac{5 \cdot 3 - 3}{6}$ is an integer.)

(b) Is the following true or false? $\forall n \in \mathbb{Z}, P(n)$. Explain.

Solution: False, because $P(1)$ is false. (i.e. $\frac{5 \cdot 1 - 3}{6}$ is not an integer.)

(3) Rewrite the following using quantifiers:

(a) There is a real number such that $1/y = y + 1$.

Solution: $\exists y \in \mathbb{R}, (1/y = y + 1)$.

(b) For every natural number z, there is a natural number w such that $z^2 < w$.

Solution: $\forall z \in \mathbb{N}, (\exists w \in \mathbb{N}, (z^2 < w))$.

(4) Let A, B be subsets of a universal set U. Rewrite the following using quantifiers involving the elements. For example, $\forall x \in U, \ldots$

(a) $A \subset B$

Solution: $(\forall x \in A, (x \in B)) \wedge (\exists x \in B, (x \notin A))$.

(b) $A \cap B = \emptyset$

Solution: $\forall x \in A, (x \notin B)$. Another way is $\forall x \in U, ((x \notin A) \vee (x \notin B))$.

(c) $A \not\subset B$

Solution: Just negate (a).

$$\begin{aligned} &\sim((\forall x \in A, (x \in B)) \wedge (\exists x \in B, (x \notin A))) \\ &\equiv (\not\forall x \in A, (x \in B)) \vee (\nexists x \in B, (x \notin A)) \\ &\equiv (\exists x \in A, (x \notin B)) \vee (\forall x \in B, (x \in A)) \end{aligned}$$

(5) Is the following sentence true or false?

$$\forall \epsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0}, |(3+\delta)^2 - 3^2| < \epsilon$$

Write its negation.

Solution: The sentence is true. Here is an argument why. It is enough to show that for any $\epsilon \in \mathbb{R}_{>0}$, there exists $\delta \in \mathbb{R}_{>0}$ such that $6\delta + \delta^2 < \epsilon$. If we pick δ such that $(7\delta < \epsilon) \wedge (7\delta^2 < \epsilon)$, then $6\delta + \delta^2 < \epsilon$, because $6\delta + \delta^2$ is between 7δ and $7\delta^2$. δ can be chosen to be any real number less than $\min(\epsilon/7, \sqrt{\epsilon/7})$. Letting $\delta = \frac{1}{2}\min(\epsilon/7, \sqrt{\epsilon/7})$ suffices.

The negation is

$$\exists \epsilon \in \mathbb{R}_{>0}, \forall \delta \in \mathbb{R}_{>0}, |(3+\delta)^2 - 3^2| \geq \epsilon$$

(6) For what values of x is the following open sentence (in x) true?

$$\forall \epsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0}, |(x+\delta)^2 - x^2| < \epsilon$$

How about this one?

$$\exists \delta \in \mathbb{R}_{>0}, \forall \epsilon \in \mathbb{R}_{>0}, |(x+\delta)^2 - x^2| < \epsilon$$

Solution: The first sentence is true for all x. For any x, one can use an argument identical to that in the previous question to find that $\delta = |\frac{1}{2}\min\left(\frac{\epsilon}{2x+1}, \sqrt{\frac{\epsilon}{2x+1}}\right)|$ satisfies the required condition.

The second sentence is false because its negation

$$\forall \delta \in \mathbb{R}_{>0}, \exists \epsilon \in \mathbb{R}_{>0}, |(x+\delta)^2 - x^2| \geq \epsilon$$

is true. For any δ, taking $\epsilon = \frac{1}{2}|\delta^2 + 2x\delta|$ satisfies the required condition.