MATH 220.201 CLASS 2 QUESTIONS - SOLUTIONS

1. Complement and Set Difference

Let U denote the universal set.

(1) Prove that $A - B = A \cap \overline{B}$.

Proof:

$$\begin{aligned} A - B &= \{x | x \in A \text{ and } x \notin B\} \\ &= \{x|x \in A\} \cap \{x|x \notin B\} \\ &= A \cap \overline{B} \end{aligned}$$

(2) What is $\overline{U}$? **Answer:** $\overline{U} = \emptyset$.

(3) (De Morgan's Laws) Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$ and $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: See the text. Note that this also holds for arbitrary unions and intersections, i.e. if I is any index set (even an infinite one!) then

$$\overline{\bigcup_{i\in I} A_i} = \bigcap_{i \in I} \overline{A_i} \qquad \overline{\bigcap_{i\in I} A_i} = \bigcup_{i \in I} \overline{A_i}$$

(4) Find three sets A, B, and C such that $(A \cup B) \cap C \neq A \cup (B \cap C)$. (Hint: Draw Venn diagrams for $(A \cup B) \cap C$ and $A \cup (B \cap C)$.)

Solution: One can show that $(A \cup B) \cap C = A \cup (B \cap C)$ occurs if and only if $A \subseteq C$. So just take sets such that $A \subsetneq C$ and you'll find that $(A \cup B) \cap C \neq A \cup (B \cap C)$.

2. Indexed Union and Intersection

(1) For each $k \in \mathbb{N}$, define $A_k \subset \mathbb{R}$ by $A_k = [\frac{1}{k+1}, \frac{1}{k}]$. What is $\bigcup_{k=1}^{\infty} A_k$?

Solution: $\bigcup_{k=1}^{\infty} A_k = (0, 1]$. This is because

• Every A_k consists entirely of positive numbers less than or equal to 1. Therefore, $\bigcup_{k=1}^{\infty} A_k$ consists only of positive numbers less than or equal to 1, so $\bigcup_{k=1}^{\infty} A_k \subseteq (0, 1]$.

• Every $x \in (0, 1]$ is contained in at least one set A_k. Therefore, $(0, 1] \subseteq \bigcup_{k=1}^{\infty} A_k$.

- $\bigcup_{k=1}^{\infty} A_k \subseteq (0,1]$ and $(0,1] \subseteq \bigcup_{k=1}^{\infty} A_k$ together imply that the two sets are equal.

(2) For each $k \in \mathbb{N}$, define $B_k \subset \mathbb{R}$ by $B_k = (-\frac{1}{k}, \frac{1}{k})$. What is $\bigcap_{k=1}^{\infty} B_k$?

Solution: $B_k = \{0\}$. This is because

- $\{0\}$ is a subset of every B_k, and therefore $\{0\} \subseteq \bigcap_{k=1}^{\infty} B_k$.
- Suppose x is any nonzero real number. Then $|x|$ is a positive number, and so there is some $k \in \mathbb{Z}$ such that $\frac{1}{k} \leq |x|$. Then, $x \notin B_k$. Therefore, $x \notin \bigcap_{k=1}^{\infty} B_k$. This holds true for every nonzero real number x, so $\bigcap_{k=1}^{\infty} B_k \subseteq \{0\}$.
- $\{0\} \subseteq \bigcap_{k=1}^{\infty} B_k$ and $\bigcap_{k=1}^{\infty} B_k \subseteq \{0\}$ together imply that the two sets are equal.

(3) Let A and B be sets. Prove that $\bigcap_{b \in B} (A - \{b\}) = A - B$.

Solution:

$$\begin{aligned}\bigcap_{b\in B}(A - \{b\}) &= \{x | x \in A - \{b\} \text{ for all } b \in B\} \\ &= \{x | x \in A \text{ and } x \neq b \text{ for all } b \in B\} \\ &= \{x | x \in A \text{ and } x \notin B\} \\ &= A - B\end{aligned}$$

3. Set Partitions and Cartesian Product

(1) List out the partitions of the set $\{1, 2, 3\}$. How many partitions are there for the set $\{1, 2, 3, 4\}$? (Can you count them without listing them out?)

Solution: There are five.

$\{\{1, 2, 3\}\}; \{\{1, 2\}, \{3\}\}; \{\{1, 3\}, \{2\}\}; \{\{2, 3\}, \{1\}\}; \{\{1\}, \{2\}, \{3\}\}$

(2) Construct a partition of $\mathbb{Z}$ into two sets.

Solution: Here are a few different ones.

$$\{\{0\}, \{\ldots, -2, -1, 1, 2, \ldots\}\}$$
$$\{\{\ldots, -3, -2, -1\}, \{0, 1, 2, 3, \ldots\}\}$$
$$\{\{\ldots, -2, 0, 2, 4, \ldots\}, \{\ldots, -3, -1, 1, 3, \ldots\}\}$$

In general, if $S \subseteq \mathbb{Z}$ is any subset of $\mathbb{Z}$, there's a partition

$$\{S, \mathbb{Z} - S\}$$

and every partition of $\mathbb{Z}$ arises in this way.

(3) Construct a partition of $\mathbb{Z}$ into three *infinite* sets.

Solution: Here's one example: consider the partition $\{S_0, S_1, S_2\}$ where[1]

$$S_0 = \{\dots, -3, 0, 3, 6, \dots\} = \{3x | x \in \mathbb{Z}\}$$
$$S_1 = \{\dots, -2, 1, 4, 7, \dots\} = \{3x + 1 | x \in \mathbb{Z}\}$$
$$S_2 = \{\dots, -1, 2, 5, 8, \dots\} = \{3x + 2 | x \in \mathbb{Z}\}$$

(4) Construct a partition of $\mathbb{Q}$ into two infinite sets.

Solution: Pick any number $r \in \mathbb{Q}$. Then we may define the two sets

$$L_r = \{x | x \in \mathbb{Q} \text{ and } x < r\}$$
$$G_r = \{x | x \in \mathbb{Q} \text{ and } x \geq r\}$$

Then $\{L_r, G_r\}$ is a partition of $\mathbb{Q}$. This works not just when r is rational, but even works when r is *irrational*![2]

(5) How many elements are in the set $\{(x, y) | (x, y) \in \mathbb{R}^2 \text{ and } x^2 + y^2 < 12\}$?

Solution: Trick question: it is an infinite set! This set is the interior (not including the boundary) of a circle with radius $\sqrt{12}$ centered at the origin.

(6) Let $S \subset \mathbb{R}^2$ be the set of points shown on the left. Write down, in terms of S, the set on the right (i.e., a reflection over the line $y = \frac{1}{2}$).

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Solution: The set on the right is obtained by reflecting each point of S over the line $y = \frac{1}{2}$. Reflecting any point (x, y) over this line gives the point $(x, 1 - y)$, and so the set on the right is

$$\{(x, 1 - y) | (x, y) \in S\}$$

[1]The numbers in S_1 are said to be '1 modulo 3' and the numbers in S_2 '2 modulo 3'. Modular arithmetic is one of the fundamental concepts in number theory, and we will return to this later in the course!

[2]When r is irrational, G_r has no smallest element. That is,

$$r \in \mathbb{R} - \mathbb{Q} \implies \{x | x \in \mathbb{Q} \text{ and } x \geq r\} = \{x | x \in \mathbb{Q} \text{ and } x > r\}$$

In this case, the partition $\{L_r, G_r\}$ is a *Dedekind cut* of the set $\mathbb{Q}$. Dedekind cuts are a way to axiomatically construct the set $\mathbb{R}$ of real numbers from the set $\mathbb{Q}$ of rational numbers.