MATH 220.201 CLASS 25 QUESTIONS

1. Prove that the sequence $\left\{\frac{4n^2-3}{5n^2-2n}\right\}$ converges to $\frac{4}{5}$.

   Proof 1. We can calculate
$$\frac{4n^2-3}{5n^2-2n}-\frac{4}{5}=\frac{5(4n^2-3)-4(5n^2-2n)}{5(5n^2-2n)}=\frac{8n-15}{25n^2-10n}$$
   We must show that for any ϵ, there is some N such that for every $n > N$, this fraction has absolutely value less than ϵ. But $|8n-15| \leq |8n|$, and as long as $n \geq 10$, we have $|25n^2-10n| \geq |24n^2|$. Therefore, as long as $n \geq 10$,
$$\left|\frac{8n-15}{25n^2-10n}\right| \leq \left|\frac{8n}{24n^2}\right| = \left|\frac{1}{3n}\right|$$
   Therefore, if one takes $n > \max(10, 1/3\epsilon)$, we will have $|\frac{4n^2-3}{5n^2-2n} - \frac{4}{5}| < \epsilon$, as desired. □

   Proof 2. Use the limit laws.
$$\lim_{n\to\infty}\frac{4n^2-3}{5n^2-2n}=\lim_{n\to\infty}\frac{4-\frac{3}{n^2}}{5-\frac{2}{n}}=\frac{4-\lim_{n\to\infty}\frac{3}{n^2}}{5-\lim_{n\to\infty}\frac{2}{n}}=\frac{4}{5}$$
   □

2. (12.11) Prove (using the definition of convergence) that if a sequence $\{s_n\}$ converges to L, then the sequence $\{s_{n^2}\}$ converges to L.

   Proof. $\{s_n\}$ converges to L, which means that for every $\epsilon > 0$, there exists some N such that if $n > N$, then $|s_n - L| < \epsilon$. If $n > N$, then $n^2 > N$ as well, because $n^2 \geq n$ for every natural number n. It therefore follows that if $n > N$, $|s_{n^2} - L| < \epsilon$. Thus, for every ϵ, the same value of N works for the sequence $\{s_{n^2}\}$ as for the sequence $\{s_n\}$. □

3. (Adapted from 2011 WT2 Q7) Let $\{b_n\}$ be a sequence defined by
$$b_1 = 2 \text{ and } b_{n+1} = \frac{b_n+\sqrt{b_n}}{2}$$
   (a) Prove that $1 < b_{n+1} < b_n$ for every $n \in \mathbb{N}$.
   (b) Prove that $\{b_n\}$ converges to 1.

Proof. (a) We prove that $1 < b_{n+1} < b_n$ by induction on n. The base case, $n = 1$, holds true because $b_2 = 1+\frac{1}{\sqrt{2}}$, which is greater than 1 and smaller than $2 = 1+1$.

Let us suppose that $1 < b_{n+1} < b_n$. We will show that $1 < b_{n+2} < b_{n+1}$. Since $1 < b_{n+1}$, it follows that

$$\begin{aligned} 1 < \sqrt{b_{n+1}} < b_{n+1} &\implies 2 < b_{n+1} + \sqrt{b_{n+1}} < 2b_{n+1} \\ &\implies 1 < \frac{b_{n+1} + \sqrt{b_{n+1}}}{2} < b_{n+1} \\ &\implies 1 < b_{n+2} < b_{n+1} \end{aligned}$$

which completes the induction.

(b) The sequence $\{b_n\}$ is decreasing and is bounded below by 1, and therefore it must converge to a limit $L \geq 1$. Let us suppose for a contradiction that $L > 1$. Then for any $\epsilon > 0$, there exists some N such that if $n > N$, $b_n < L + \epsilon$. Then this implies that

$$\begin{aligned} b_{n+1} = \frac{b_n + \sqrt{b_n}}{2} &< \frac{L + \epsilon + \sqrt{L+\epsilon}}{2} \\ &= \frac{L + \epsilon + \sqrt{L(1+\frac{\epsilon}{L})}}{2} \\ &< \frac{L + \epsilon + \sqrt{L(1+\frac{\epsilon}{L}+\frac{\epsilon^2}{4L^2})}}{2} \\ &= \frac{L + \epsilon + \sqrt{L(1+\frac{\epsilon}{2L})^2}}{2} \\ &= \frac{L + \epsilon + (1+\frac{\epsilon}{2L})\sqrt{L}}{2} \\ &= \frac{L+\sqrt{L}}{2} + \frac{\epsilon}{2}\left(1 + \frac{1}{\sqrt{L}}\right) \\ &< \frac{L+\sqrt{L}}{2} + \epsilon \end{aligned}$$

Pick any positive number $\epsilon < \frac{L-\sqrt{L}}{4}$. Then it will be the case that $\frac{L+\sqrt{L}}{2} + \epsilon < L - \epsilon$, and thus $b_{n+1} < L - \epsilon$. But this contradicts the assumption that $|b_{n+1} - L| < \epsilon$! Therefore, it was impossible that $L > 1$. □

4. (2013 WT1, Q8) Prove that the function $f : (-1, 1] - \{0\} \to \mathbb{R}$ given by $f(x) = x - \frac{1}{x}$ is a bijection.

Proof. First, we prove the function is injective. Suppose that $x - \frac{1}{x} = y - \frac{1}{y}$ for some numbers $x, y \in (-1, 1] - \{0\}$. Then

$$\begin{aligned} x - y &= \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy} \\ &\implies (x-y)(xy-1) = 0 \end{aligned}$$

It's impossible to have $xy = 1$ by absolute value considerations, unless $x = y = 1$. Thus, we have $x - y = 0$, which implies $x = y$. So the function is injective.

Now we prove it's surjective. Let r be any real number. Then we solve the equation $x - \frac{1}{x} = r$

$$x^2 - rx - 1 = 0 \iff x = \frac{r \pm \sqrt{r^2+4}}{2}$$

If r is zero, then the $+$ root is 1. If r is positive, the - root lies in $(-1, 1] - \{0\}$ by difference of squares:

$$\frac{r - \sqrt{r^2+4}}{2} = \frac{-4}{2(r + \sqrt{r^2+4})} = -\frac{2}{r + \sqrt{r^2+4}} > -\frac{2}{0 + \sqrt{0+4}} = -1$$

If r is negative, the $+$ root lies in $(-1, 1] - \{0\}$ by a similar argument. □

5. Give an example of
   - A function $f : \mathbb{Z} \to \mathbb{Z}$ which is injective, but not surjective.

     Solution: Try the function $f(n) = 2n$.
   - A function $g : \mathbb{Z} \to \mathbb{Z}$ which is surjective, but not injective.

     Solution: Try the function $g(n) = \lfloor n/2 \rfloor$.
   - Prove that for any such functions f and g, $f \circ g$ cannot possibly be bijective.

     Solution: Because g is not injective, there are two distinct integers x and y such that $g(x) = g(y)$. Then $f(g(x)) = f(g(y))$, so $f \circ g$ is not injective. So it can't possibly be bijective. (You can show $f \circ g$ is not surjective, either.)
   - Give such examples with the property that $g \circ f$ is bijective.

     Solution: The the examples I gave above, $f(n) = 2n$ and $g(n) = \lfloor n/2 \rfloor$. Then $g(f(n)) = n$.