MATH 220.201 CLASS 23 QUESTIONS

Definition 0.1. *Let* $\{a_n\} = a_1, a_2, a_3, \ldots$ *be a sequence of real numbers, and let* L *be another real number.* $\{a_n\}$ *is said to* **converge to** L *if the following holds:*

For every real number $\epsilon > 0$*, there exists some* $N \in \mathbb{N}$*, such that for every natural number* $n > N$*,* $|a_n - L| < \epsilon$*.*

1. Prove that the sequence $\{\frac{n}{2n+1}\}$ converges.

   Scrapwork: Intuitively, as $n \to \infty$, $\frac{n}{2n+1} \to 1/2$. To prove this, we need to show that for any ϵ, there exists some N such that if $n > N$, the inequality $|\frac{n}{2n+1} - \frac{1}{2}| < \epsilon$ holds true. We have to figure out how big n must be (in terms of ϵ) to make this inequality happen. Some simple algebra shows that

   $$|\frac{n}{2n+1} - \frac{1}{2}| = |\frac{2n - (2n+1)}{2(2n+1)}| = |\frac{-1}{4n+2}| = \frac{1}{4n+2}$$

   so we must have $\frac{1}{4n+2} < \epsilon$. This happens iff $n > \frac{1}{4\epsilon} - \frac{1}{2}$. So if we take $N = \lceil 1/4\epsilon \rceil$, then this inequality will always hold true if $n > N$.

   Proof. We'll show that this sequence converges to 1/2. Let ϵ be any positive real number. Then we claim that for any $n > \lceil 1/4\epsilon \rceil$,

   $$|\frac{n}{2n+1} - \frac{1}{2}| < \epsilon$$

   that is, we may take $N = \lceil 1/4\epsilon \rceil$ This is because

   $$|\frac{n}{2n+1} - \frac{1}{2}| = |\frac{-1}{4n+2}| = \frac{1}{4n+2} < \frac{1}{1/\epsilon} = \epsilon$$

   This completes the proof. □

2. Prove that the sequence $\{n\}$ diverges.

   Proof. Suppose, for a contradiction, that this sequence has some limit L. Then for any ϵ, there is some N such that for any $n > N$, $|n - L| < \epsilon$. Pick $\epsilon = 1/3$, and let N be the number satisfying the condition for this ϵ. Then $|(N+1) - L| < 1/3$ and $|(N+2) - L| < 1/3$. By the Triangle Inequality,

   $$|(N+2) - (N+1)| \leq |(N+2) - L| + |L - (N+1)|$$

   But the left side equals 1, while the right side is less than 2/3. This is a contradiction! Thus, no such L exists. □

3. Does the sequence $\{\frac{2^n - 1}{2^n} + \frac{(-1)^n}{n^2}\}$ converge?

4. Suppose that $\{a_n\}$ is a sequence of real numbers which converges to L, and suppose that $a_n > 1$ for every n. Is it true that $\{1/a_n\}$ converges to $1/L$?

These are both answered by the proof of the following proposition.

Proposition 0.2. *Let $a_n \to a$ and $b_n \to b$ be two convergent sequences. Then*
- $a_n + b_n \to a + b$
- $a_n b_n \to ab$
- *For any* $c \in \mathbb{R}$, $ca_n \to ca$
- $1/b_n \to 1/b$ *as long as* $b_n \neq 0, b \neq 0$

Proof.
- For any ϵ, $\epsilon/2$ is another positive real number, and so there exist natural numbers M, N such that if $m > M$, $|a_m - a| < \epsilon/2$ and if $n > N$, $|b_n - b| < \epsilon/2$. Therefore, if $k > \max(M, N)$,
$$|a_k - a| + |b_k - b| < \epsilon$$
By the Triangle Inequality, $|(a_k + b_k) - (a + b)| < |a_k - a| + |b_k - b|$, and so $|(a_k + b_k) - (a + b)| < \epsilon$. Thus, $\max(M, N)$ satisfies the required condition for the sequence $a_k + b_k$.

- Pick any ϵ. First let's notice that
$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |(a_n - a)b_n| + |a(b_n - b)|$$
Thus, if we want to have $|a_n b_n - ab| < \epsilon$, it suffices to have $|(a_n - a)b_n| < \epsilon/2$ and $|a(b_n - b)| < \epsilon/2$. We know that for n larger than some N_1, $|b_n - b| < b/2$ (i.e. b_n is between $b/2$ and $3b/2$) and for n larger than some N_2, $|a_n - a| < \epsilon/4b$. Therefore, for n larger than $\max(N_1, N_2)$, these both become true, which then implies that
$$|(a_n - a)b_n| < (\epsilon/4b)(3b/2) < \epsilon/2$$
Similarly, for n larger than some N_3, $|b_n - b| < \epsilon/2a$ and thus
$$|a(b_n - b)| < \epsilon/2$$
So for $n > \max(N_1, N_2, N_3)$, both of the displayed inequalities are true, and thus $|a_n b_n - ab| < \epsilon$.

- For any $\epsilon > 0$, ϵ/c is another positive real number, and so there is some $N(\epsilon/c)$ such that if $n > N(\epsilon/c)$, then $|a_n - a| < \epsilon/c$. This means that $|ca_n - ca| < \epsilon$. Thus, $N(\epsilon/c)$ is the value of N required for the sequence ca_n.

- Pick any ϵ. For n larger than some N_1, $|b_n - b| < \frac{b^2}{2} \cdot \epsilon$. This means that
$$|bb_n \left(\frac{1}{b} - \frac{1}{b_n} \right) | < \frac{b^2}{2} \cdot \epsilon$$
For n larger than some N_2, $|b_n - b| < b/2$, in which case
$$|\frac{b^2}{2} \left(\frac{1}{b} - \frac{1}{b_n} \right) | < |bb_n \left(\frac{1}{b} - \frac{1}{b_n} \right) |$$

Thus, for $n > \max(N_1, N_2)$, $|\frac{1}{b} - \frac{1}{b_n}| < \epsilon$, as desired. □

5. Prove that the sequence $\{a_n\}$ defined by $a_1 = 1$ and $a_{n+1} = a_n + \frac{1}{a_n}$ diverges. (Hint: Suppose that it converges, and use proof by contradiction.)

Proof. I'm going to save this for next class. Think about it! □

6. Prove that the sequence $\{a_n\}$ defined by $a_1 = 1$ and $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$ converges to $\sqrt{2}$. (Hint: Define a new sequence $\{b_n\}$ by $b_n = a_n - \sqrt{2}$. Now find a recursive formula for this new sequence and use that to show that it converges to 0.)

Proof. Define the sequence

$$b_n = a_n - \sqrt{2}$$

Then we can find a recursive formula

$$\begin{aligned} b_{n+1} = a_{n+1} - \sqrt{2} &= \frac{a_n}{2} + \frac{1}{a_n} - \sqrt{2} \\ &= \frac{b_n + \sqrt{2}}{2} + \frac{1}{b_n + \sqrt{2}} - \sqrt{2} \\ &= \frac{(b_n + \sqrt{2})^2 + 2 - 2(b_n + \sqrt{2})\sqrt{2}}{2(b_2 + \sqrt{2})} \\ &= \frac{b_n^2}{2(b_n + \sqrt{2})} \end{aligned}$$

We'll first show by induction that $1 - \sqrt{2} \le b_n \le \sqrt{2} - 1$ for every $n \in \mathbb{N}$. Notice that $b_1 = 1 - \sqrt{2}$ - this is the base case. And for any given n, if $1 - \sqrt{2} \le b_n \le \sqrt{2} - 1$, then

$$0 \le \frac{b_n^2}{2(b_n + \sqrt{2})} < \frac{b_n^2}{2} < \frac{|b_n|}{2}$$

and therefore, $|b_{n+1}| < |b_n|/2$ $(\star)$, which then implies that $1 - \sqrt{2} \le b_{n+1} \le \sqrt{2} - 1$. So $1 - \sqrt{2} \le b_n \le \sqrt{2} - 1$ for every $n \in \mathbb{N}$.

Note that we also showed $|b_{n+1}| < |b_n|/2$ for every $n \in \mathbb{N}$ (marked with a $(\star)$). From this, it is clear by induction that $|b_n| < 1/2^n$, and so $b_n \to 0$. Since $b_n = a_n - \sqrt{2}$, it follows that $a_n \to \sqrt{2}$. □