# MATH 220.201 CLASS 22 QUESTIONS

1. Let $A$ and $B$ be sets, and suppose there is an injective function $f : A \to B$. Can you construct a surjective function $g : B \to A$?

   **Solution:** Yes we can, so long as $A$ is nonempty. Pick some fixed element $x$ in $A$. The fact that $f : A \to B$ is injective means that for every $b \in B$, there is at most one $a \in A$ such that $f(a) = b$. Then we can construct a surjective function $g : B \to A$ defined by the property that
   - If $b \in \mathrm{im}(f)$, then let $g(b)$ be the unique element such that $f(g(b)) = b$.
   - If $b \notin \mathrm{im}(f)$, then let $g(b) = x$.

   Note that this construction wouldn't work if $A$ were empty. For example, if $A = \emptyset$ and $B = \{1, 2, 3\}$, then there are $3^0 = 1$ functions from $A$ to $B$, and $0^3 = 0$ functions from $B$ to $A$.

2. Let $A$ and $B$ be sets, and suppose there is a surjective function $g : B \to A$. Can you construct an injective function $f : A \to B$?

   **Solution:** Yes you can, but doing so requires the *Axiom of Choice*. The fact that $g : A \to B$ is surjective means that for every $b \in B$, there is at least one $a \in A$ such that $g(a) = b$. So for each $b \in B$, choose[1] some element $a \in g^{-1}(b)$ and declare that $f(b)$ is equal to that $a$. The result is an injective function, because $f(b)$ satisfies the property that $g(f(b)) = b$ - therefore if $f(b) = f(b')$ then $g(f(b)) = g(f(b'))$ which means $b = b'$.

3. Prove that
$$|2^{\mathbb{N}}| = |2^{\mathbb{N}} \times 2^{\mathbb{N}}|$$
   by finding an explicit bijection. (Remember that $2^A$ is the set of functions from $A$ to $\{0, 1\}$.)

   *Proof.* In this proof, 'function' will be used to mean a function $\mathbb{N} \to \{0, 1\}$. Given any function $f$, define functions $f_{\text{odd}}$ and $f_{\text{even}}$ by
$$f_{\text{odd}}(n) = f(2n - 1) \qquad\qquad f_{\text{even}}(n) = f(2n)$$
   Then we have
$$H : 2^{\mathbb{N}} \to 2^{\mathbb{N}} \times 2^{\mathbb{N}}$$
$$f \mapsto (f_{\text{odd}}, f_{\text{even}})$$
   If $f$ and $g$ are functions such that $(f_{\text{odd}}, f_{\text{even}}) = (g_{\text{odd}}, g_{\text{even}})$, then $f(2n-1) = g(2n-1)$ for every $n$ and $f(2n) = g(2n)$ for every $n$. It then follows that $f = g$. Thus, $H$ is injective.

[1] This is the step which requires the Axiom of Choice.

If $(f, g)$ is any pair of functions, then one may consider the function $h$ defined by

$$h(n) = \begin{cases} f\left(\frac{n+1}{2}\right) & n \text{ odd} \\ g\left(\frac{n}{2}\right) & n \text{ even} \end{cases}$$

and one sees that $h_{\text{odd}} = f$ and $h_{\text{even}} = g$, i.e. $H(h) = (f, g)$. Thus, $H$ is surjective. □

**Note:** A corollary of this fact is that $|\mathbb{R}| = |\mathbb{R} \times \mathbb{R}|$.

4. Let $A, B$, and $C$ be any three sets, and let $A, B$ be disjoint. Prove that $|C^{A \cup B}| = |C^A \times C^B|$, or written another way,

$$|\text{Fun}(A \cup B, C)| = |\text{Fun}(A, C) \times \text{Fun}(B, C)|$$

*Proof.* Given a function $f : A \cup B \to C$, one obtains functions $f_A : A \to C$ and $f_B : B \to C$ by restricting the action of $f$. Thus, we have

$$\Phi : \text{Fun}(A \cup B, C) \to \text{Fun}(A, C) \times \text{Fun}(B, C)$$

$$f \mapsto (f_A, f_B)$$

Similarly, if one has functions $g : A \to C$ and $h : B \to C$, one obtains a function $g \cup h : A \cup B \to C$ defined by

$$(g \cup h)(x) = \begin{cases} g(x) & x \in A \\ h(x) & x \in B \end{cases}$$

Thus, we have

$$\Theta : \text{Fun}(A, C) \times \text{Fun}(B, C) \to \text{Fun}(A \cup B, C)$$

$$(g, h) \mapsto g \cup h$$

It's easy to check that $\Theta$ and $\Phi$ are inverses of one another, and so both are bijections. □

5. (More challenging) Prove that $|C^{A \times B}| = |(C^A)^B|$, or written another way,

$$|\text{Fun}(A \times B, C)| = |\text{Fun}(B, \text{Fun}(A, C))|$$

*Proof.* First, here's a function

$$\Phi : \text{Fun}(B, \text{Fun}(A, C)) \to \text{Fun}(A \times B, C)$$

Suppose that we have a function $f : B \to \text{Fun}(A, C)$. Then one obtains a function $\Phi(f) : A \times B \to C$ by

$$\Phi(f)(a, b) = (f(b))(a)$$

Next, here's a function

$$\Theta : \text{Fun}(A \times B, C) \to \text{Fun}(B, \text{Fun}(A, C))$$

Suppose that we have a function $g : A \times B \to C$. Then we obtain a function $\Theta(g) : B \to \mathrm{Fun}(A, C)$ by

$$(\Theta(g)(b))(a) = g(a, b)$$

It's fairly easy to check that $\Phi$ and $\Theta$ are inverses of one another. $\square$