MATH 220.201 CLASS 20 QUESTIONS

1. (a) Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Prove that there exists a surjective function $\mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$.

   Solution: There's such a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}^+$ given by

   $$f((a, b)) = a/b$$

   It is surjective because every positive rational number can be written as a fraction with positive numerator and denominator.

   (b) Deduce that there exists a surjective function $\mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$.

   Solution: One can again define a function $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ by $f((a, b)) = a/b$. It is well-defined as $b \neq 0$. It is surjective because every rational number can be written as a fraction (with top and bottom both integers) with positive denominator.

   (c) Show that there is an injective function $\mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$.

   Solution: Any rational number q can be written *uniquely* in the form $q = \frac{a}{b}$ where $b > 0$ and a, b have no common factors. Because this is unique, we get a function $\mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$. The function is clearly injective because two different rational numbers can't be given by the same fraction.

2. For any integer k, let $k^{\mathbb{N}}$ denote the set of functions $\mathbb{N} \to \{1, 2, \dots, k\}$. We'll prove in class today that $2^{\mathbb{N}}$ is *uncountable*.

   (a) Explain why there is a bijection between $2^{\mathbb{N}}$ and $\mathcal{P}(N)$, the *power set* of $\mathbb{N}$.

   Solution: Consider the function $\theta : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ given as follows. Suppose that $f \in 2^{\mathbb{N}}$. Then $f^{-1}(1)$ is a subset of $\mathbb{N}$, and we let $\theta(f) = f^{-1}(1)$.
   In the other direction, consider the function $\varphi : \mathcal{P}(\mathbb{N}) \to 2^{\mathcal{N}}$ given as follows. Suppose that $S \in \mathcal{P}(\mathbb{N})$. Then one may define a function $\varphi(S); \mathcal{N} \to \{1, 2\}$ by

   $$(\varphi(S))(n) = \begin{cases} 1 & n \in S \\ 2 & n \notin S \end{cases}$$

   Let $\varphi(S)$ be this function.
   We have defined functions $\theta : 2^{\mathbb{N}} \to \mathcal{P}(\mathbb{N})$ and $\varphi : \mathcal{P}(\mathbb{N}) \to 2^{\mathcal{N}}$. We will now show they are inverse functions. First, we show that $\varphi(\theta(f)) = f$ for any

$f \in 2^{\mathbb{N}}$. For any $n \in \mathbb{N}$,

$$(\varphi(\theta(f)))(n) = \begin{cases} 1 & n \in f^{-1}(1) \\ 2 & n \notin f^{-1}(1) \end{cases}$$

It is clear that the function $\varphi(\theta(f))$ is equal to f. Now we check that $\theta(\varphi(S)) = S$. $\theta(\varphi(S)) = \varphi(S)^{-1}(1)$ which equals S by the definition of φ. Thus, $\theta(\varphi(S)) = S$ as desired.

(b) Show that there is a surjective function $10^{\mathbb{N}} \to [0,1]$, where $[0,1]$ is the interval of real numbers x such that $0 \leq x \leq 1$.

Solution: We'll instead consider functions from $\mathbb{N}$ to the set $\{0,1,2,\ldots,9\}$, as that set is bijective with $10^{\mathbb{N}}$. Consider the function which sends a function

$$f : \mathbb{N} \to \{0,1,2,\ldots,9\}$$

to the sum $\sum_{n=1}^{\infty} \frac{f(n)}{10^n}$. In other words, it sends the function $f(-)$ to the decimal number

$$f(-) \mapsto 0.f(1)f(2)f(3)f(4)\ldots$$

This function is surjective because every real number in $[0,1]$ has a decimal representation of this form (for the number 1, note that 1=0.999...).

(c) Show that, for any positive integer k, there is a surjective function $k^{\mathbb{N}} \to [0,1]$.

Solution: Do the same thing but base k instead of base 10.

$$f(-) \mapsto \sum_{n=1}^{\infty} \frac{f(n)}{k^n}$$

3. Let $S \subset \mathcal{P}(\mathbb{N})$ be the set of *finite* subsets of $\mathbb{N}$. Prove that S is denumerable.
Solution 1:

Lemma 0.1. *Let $A_1, A_2, A_3, \ldots$ be a sequence of disjoint, denumerable sets. Then $A_1 \cup A_2 \cup A_3 \cup \cdots$ is denumerable.*

Proof of Lemma. The assumption is that there is a bijection $f_n : \mathbb{N} \to A_n$ for every $n \in \mathbb{N}$. Then there is a bijection

$$\mathbb{N} \times \mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$$

which maps $\{n\}\times\mathbb{N}$ bijectively to A_n. Compose this with the bijection[1] $\mathbb{N}\to\mathbb{N}\times\mathbb{N}$ and we obtain a bijection $\mathbb{N}\to\bigcup_{n=1}^{\infty} A_n$. □

Lemma 0.2. *Let $S_1, S_2, S_3, \ldots$ be a sequence of nonempty, disjoint, finite sets. Then $S_1\cup S_2\cup S_3\cup\cdots$ is denumerable.*

Proof. Let a_n be the size of set S_n for each n. Then we can form bijections

$$\begin{gathered}\{1,2,\ldots,a_1\}\to S_1\\ \{a_1+1,a_1+2,\ldots,a_1+a_2\}\to S_2\\ \{a_1+a_2+1,a_1+a_2+2,\ldots,a_1+a_2+a_3\}\to S_3\\ \vdots\end{gathered}$$

Putting these together gives a bijection from $\mathbb{N}$ to $S_1\cup S_2\cup S_3\cup\cdots$. □

Proof of Question 3. For each n, let A_n denote the set of n-element subsets of $\mathbb{N}$. By the first Lemma, it suffices to show that, for any n, A_n is denumerable.

For every natural number $m\geq n$, let B_m be the set of n-element subsets of $\mathbb{N}$ with largest element equal to m. So for example, B_n consists of a single element, B_{n+1} consists of $n+1$ elements, and so on. Then A_n is the disjoint union of $B_n, B_{n+1}, B_{n+2}, \ldots$ and each B_m is finite and nonempty. By the second Lemma, it follows that A_n is denumerable, as desired. □

Solution 2: There is a function

$$f: S\to\mathbb{Q}$$

which is given by $f(S)=\sum_{n\in S}\frac{1}{10^n}$. This is easily seen to be an injective function, because no rational number has two different finite-length decimal representations. Thus, $|S|\leq|\mathbb{Q}|$. Since $\mathbb{Q}$ is denumerable, and S is an infinite set, S is also denumerable.

[1]We constructed this in a previous class, but for completeness I will describe it here. The inverse bijection $f:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ is given by

$$f((i,j))=\frac{(i+j)(i+j+1)}{2}-i+1$$

Another possible bijection $g:\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ is by the formula

$$g((i,j))=2^{i-1}(2j-1)$$