MATH 220.201 CLASS 19 SOLUTIONS

1. Let $f : A \to B$ and $g : C \to D$ be functions. Then one can form the function

$$f \times g : A \times C \to B \times D$$

(a) Prove that if f is injective and g is injective, then $f \times g$ is injective.

Proof. Suppose that (a, c) and (a', c') are elements of $A \times C$ such that $(f(a), g(c)) = (f(a'), g(c'))$. Then $f(a) = f(a')$ and $g(c) = g(c')$. Because f is injective, it follows that $a = a'$, and because g is injective, it follows that $c = c'$. Then $(a, c) = (a', c')$.
Thus, we have proven that $f \times g$ is injective. □

(b) The same if both are surjective.

Proof. Suppose that (b, d) is any element of $B \times D$. Because f is surjective, there is some $a \in A$ such that $f(a) = b$, and because g is surjective, there is some $c \in C$ such that $g(c) = d$. Then by definition, $f \times g$ sends (a, c) to (b, d).
So we have proven that $f \times g$ is surjective. □

(c) Come up with an example where f is injective, g is surjective, and $f \times g$ is neither.

Solution: Let $f : \{0\} \to \{0, 1\}$ be defined by $f(0) = 0$, and let $g : \{0, 1\} \to \{0\}$ be defined by $g(0) = g(1) = 0$. Then $f \times g : \{(0, 0), (0, 1)\} \to \{(0, 0), (1, 0)\}$ sends both $(0, 0)$ and $(0, 1)$ to $(0, 0)$.

(d) How about an example where f is not bijective, but $f \times g$ is? (Hint: use $\emptyset$.)

Solution: Let $f : A \to B$ be any function, and let $g : \emptyset \to \emptyset$ be the trivial function consisting of no ordered pairs (check - this is a function!). Then $f \times g : A \times \emptyset \to B \times \emptyset$. But any set time the empty set is the empty set. So $f \times g$ is just the trivial function $\emptyset \to \emptyset$ again.

2. Let A, B, and C be sets. Suppose that $g : A \to B$, $h : A \to B$, and $f : B \to C$ are functions with the property that $f \circ g = f \circ h$.

(a) Provide an example of the situation above which shows that g does not necessarily have to equal h.

Solution: Let $A = \{0\}$, $B = \{1, 2\}$, and $g : A \to B$ be defined by $g(0) = 1$, $h : A \to B$ be defined by $h(0) = 2$. Let $C = \{3\}$ and let $f : B \to C$ be defined by $f(1) = f(2) = 3$. Then $f \circ g$ and $f \circ h$ both send 0 to 3, and thus are equal. But g and h are unequal functions.

(b) Prove that if f is injective, then $g = h$.

Proof. We must show that for any $a \in A$, $g(a) = h(a)$. For any $a \in A$,

$$(f \circ g)(a) = (f \circ h)(a)$$
$$\implies f(g(a)) = f(h(a))$$

Because f is injective, $g(a) = h(a)$. Thus, we have proven that $g = h$. □