MATH 220.201 CLASS 18 QUESTIONS

1. For each of the following pairs of sets A, B, determine whether there are functions from A to B which are one-to-one (injective), onto (surjective), or both (bijective). Do the same with functions from B to A.

   Note: For each of these cases, there exist plenty of functions $A \to B$ which are neither injective nor surjective.

   (a) $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, 8, 9\}$.

   Solution: There exist plenty of surjective functions $f : A \to B$ (if you are curious, there are 240 of them). One example is given by $f(1) = 6, f(2) = 7, f(3) = 8, f(4) = 9, f(5) = 9$. Written as ordered pairs, this is $\{(1, 6), (2, 7), (3, 8), (4, 9), (5, 9)\}$.
   There are plenty of injective functions $g : B \to A$ (if you are curious, there are 120 of them). One example is given by $g(6) = 1, g(7) = 2, g(8) = 3, g(9) = 4$. Written as ordered pairs, this is $\{(6, 1), (7, 2), (8, 3), (9, 4)\}$.

   (b) $A = \mathbb{N} = \{1, 2, 3, \ldots\}$ and $B = \{2n : n \in \mathbb{N}\} = \{2, 4, 6, 8, \ldots\}$.

   Solution: There are plenty of bijective functions between these two sets. One such function $f : A \to B$ is given by $f(1) = 2, f(2) = 4, f(3) = 6, f(4) = 8, \ldots$, and in general $f(n) = 2n$ for each $n \in A$.

   (c) $A = \mathbb{N}$ and $B = \{a + b\sqrt{2} : a \in \mathbb{N}, b \in \{0, 1, 2\}\}$.

   Solution: Again, there are plenty of bijective functions from A to B. Here is an example. (I've drawn double-headed arrows to indicate that the function has an inverse.)

$$\begin{array}{lll} 1 \longleftrightarrow 1 & 2 \longleftrightarrow 1+\sqrt{2} & 3 \longleftrightarrow 1+2\sqrt{2} \\ 4 \longleftrightarrow 2 & 5 \longleftrightarrow 2+\sqrt{2} & 6 \longleftrightarrow 2+2\sqrt{2} \\ 7 \longleftrightarrow 3 & 8 \longleftrightarrow 3+\sqrt{2} & 9 \longleftrightarrow 3+2\sqrt{2} \\ & \vdots & \end{array}$$

   (d) $A = \mathbb{N}$ and $B = \mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

   Solution: Again, there is a bijection. Here's a way to reorder the elements of $\mathbb{Z}$ so that this becomes clear

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$$

(e) $A = \mathbb{N}$, $B = \{2, 3, 5, 7, 11, \ldots\}$ is the set of prime numbers.

Solution: Even in this case, the two functions are in bijective correspondence. In fact, here is a more general fact.

Theorem 0.1. *Let S be any infinite subset of $\mathbb{N}$. Then there is a bijection $f : \mathbb{N} \to S$.*

Proof. Define a sequence of elements $x_1, x_2, x_3, \ldots \in S$ by the property that

$$x_n = \text{the least element of } (S - \{x_1, \ldots, x_{n-1}\})$$

For each n, $S - \{x_1, \ldots, x_{n-1}\}$ is a nonempty[1] subset of $\mathbb{N}$, and so because $\mathbb{N}$ is well-ordered, $S - \{x_1, \ldots, x_{n-1}\}$ has a least element. Thus, x_n is well-defined for each n.
This sequence $x_1, x_2, x_3, \ldots$ defines a function $f : \mathbb{N} \to S$ by the formula $f(n) = x_n$. We claim this function is both injective and surjective, hence bijective.

- Injectivity: for any two natural numbers m, n with $m < n$, we have $x_m < x_n$ by definition. Hence, the function is injective.
- Surjectivity: Suppose, for a contradiction, that there is some element $x \in S$ which is not in the image of this function - i.e not equal to any x_i. Since S is a subset of $\mathbb{N}$, $x \in \mathbb{N}$. Therefore, there are only finitely many elements of $\mathbb{N}$ which are less than x. It follows that there is some n such that $x_n > x$. But this contradicts the definition of x_n as the least element of $S - \{x_1, \ldots, x_{n-1}\}$, because x is an element of $S - \{x_1, \ldots, x_{n-1}\}$. Therefore, we have reached a contradiction, and so no such x exists. Therefore, the function is surjective.

□

[2]

2. Can you come up with a rigorous definition of what it means for a set to have 'size n'?

Definition 0.2. *A set S has size n if there exists a bijection $\{1, 2, \ldots, n\} \to S$.*

3. What about what it means for a set to be 'infinite'?

Definition 0.3. *A set S is infinite if, for every $n \in \mathbb{N}$, there does not exist any bijection $\{1, 2, \ldots, n\} \to S$.*

[1] Because S is infinite

[2] It's not true if S is an ordered set in bijection with $\mathbb{N}$, then you can always construct the bijection $\mathbb{N} \to S$ in the way described above. See *ordinal numbers* if you are curious.

4. Let m and n be two positive integers such that $m \leq n$, and suppose that S is a set and there's an injection $\{1, \ldots, n\} \to S$. Prove that if there is an injection $S \to \{1, \ldots, m\}$, then $m = n$.

Proof. $n \leq |S| \leq m$. So $n \leq m$ and $m \leq n$, which implies $m = n$. □

5. Let S be a set and suppose that there is a bijection $f : \mathbb{N} \to S$. Prove that if T is any infinite subset of S, then there is a bijection $S \to T$.

Proof. f^{-1} is a bijection from S to $\mathbb{N}$. We may consider the set $f^{-1}(T) \subset \mathbb{N}$. By the theorem proved in 1(e), there exists a bijective function $g : \mathbb{N} \to f^{-1}(T)$. Then consider the function $f \circ g \circ f^{-1} : S \to T$. The following diagram may make it easier to visualize.

$$\begin{array}{ccc} \mathbb{N} & \xleftarrow{f^{-1}} & S \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f \circ g \circ f^{-1}} \\ f^{-1}(T) & \xrightarrow{f} & T \end{array}$$

It is a composition of three bijections, and therefore it is a bijection. □