MATH 220.201 CLASS 16 QUESTIONS

1. **Proofs**
   (a) Prove that $5 \mid 3^{4n+1} + 2$ for any nonnegative integer n.

   Proof 1. We prove the result by induction on n. The base case ($n = 0$) is to show that $5 \mid 3^1 + 2$. This is clear.
   Now the inductive step. Suppose that $5 \mid 3^{4n+1} + 2$. Then there is some integer k such that $3^{4n+1} + 2 = 5k$. Then

   $$3^{4n+1} = 5k - 2 \implies 3^{4n+5} = 81(5k-2) = 405k - 162$$
   $$\implies 3^{4n+5} + 2 = 405k - 160 = 5(81k - 32)$$

   Therefore, $5 \mid 3^{4n+5} + 2$, which completes the induction. □

   Proof 2.

   $$3^{4n+1} = 3^{4n} \cdot 3^1 = 81^n \cdot 3 \equiv 1^n \cdot 3 \pmod 5 \equiv 3 \pmod 5 \equiv -2 \pmod 5$$

   Thus $5 \mid 3^{4n+1} + 2$. □

   (b) Prove that there exists a positive integer N with the following property: every odd integer $n \geq N$ can be written in the form $n = 3a + 5b + 7c$ for some positive integers a, b, c.

   Proof. We prove that every odd integer $n \geq 21$ can be written in the form $n = 3a + 5b + 7c$ for some positive integers a, b, c. Let $P(n)$ be the property that n can be written in this form. We use induction, showing that $P(21), P(23), P(25)$ are all true as the base case, and showing $P(n) \implies P(n+6)$ as the inductive step.

   - **Base case:**

   $$21 = 3 \cdot 3 + 5 \cdot 1 + 7 \cdot 1$$
   $$23 = 3 \cdot 2 + 5 \cdot 2 + 7 \cdot 1$$
   $$25 = 3 \cdot 1 + 5 \cdot 3 + 7 \cdot 1$$

   - **Inductive Step:** Suppose that $P(n)$ is true, i.e. $n = 3a + 5b + 7c$ where a, b, c are some positive integers. Then $n+6 = 3(a+2) + 5b + 7c$. Since $a+2, b, c$ are all positive integers, it follows that $P(n+6)$ is true. This completes the induction.

   □

   (c) Prove that if a and b are distinct natural numbers such that $\sqrt{a}$ and $\sqrt{b}$ are both irrational, then $\sqrt{a} + \sqrt{b}$ is also irrational.

Proof. Suppose for a contradiction that $\sqrt{a}+\sqrt{b}$ is rational. Then $(\sqrt{a}+\sqrt{b})^2$ is rational. Write $\sqrt{a}+\sqrt{b}=r$. Then

$$\sqrt{a}=r-\sqrt{b} \implies a=(r-\sqrt{b})^2=r^2-2r\sqrt{b}+b$$
$$\implies \sqrt{b}=\frac{r^2+b-a}{2r}$$

Since $\sqrt{a}$ and $\sqrt{b}$ are both positive, $\neq 0$, so this is valid. Thus, $\sqrt{b}$ is rational, which contradicts the assumption that $\sqrt{b}$ was irrational! Therefore, $\sqrt{a}+\sqrt{b}$ is irrational. □

(d) Recall that the Fibonacci sequence is defined by $F_1=1, F_2=1$, and $F_n = F_{n-1}+F_{n-2}$ for $n \geq 3$. Prove that $2^n \geq F_{n+3}$ for $n \geq 3$.

Proof. We prove the result by induction on n.

- **Base case:** When $n=3$, we have $2^3=8$ and $F_6=8$. So $2^3 \geq F_6$.
- **Inductive step:** Suppose that $2^n \geq F_{n+3}$ for some $n \geq 3$. Then

$$2^{n+1}=2^n+2^n \geq F_{n+3}+F_{n+3}=F_{n+3}+(F_{n+2}+F_{n+1})$$

Since F_{n+1} is positive,

$$F_{n+3}+F_{n+2}+F_{n+1} > F_{n+3}+F_{n+2}=F_{n+4}$$

Therefore, $2^{n+1} \geq F_{n+4}$.

□

(e) Let $a_1=1, a_2=2$, and $a_n=\sum_{i=1}^{n-1}(i-1)a_i$ for $n \geq 3$. Prove that $a_n=(n-1)!$ for $n \geq 3$.

Proof. We use induction on n.

- **Base case:** When $n=3$, $a_3=0\cdot a_1+1\cdot a_2=2$ which equals $(3-1)!$.
- **Inductive step:** Suppose that $a_n=(n-1)!$. Then

$$\begin{aligned}a_{n+1}=\sum_{i=1}^{n}(i-1)a_i &= \left(\sum_{i=1}^{n-1}(i-1)a_i\right)+(n-1)a_n\\ &= a_n+(n-1)a_n\\ &= na_n\\ &= n\cdot(n-1)!\\ &= n!\end{aligned}$$

□

(f) Prove that every positive integer can be written in the form $2^a b$ where a is a nonnegative integer and b is an odd integer.

Proof. Suppose, for a contradiction, that there are positive integers which cannot be written in the form given. Let n be the smallest such positive integer. Either n is odd or n is even. n cannot be odd, because it if is, then take $a = 0$ and $b = n$, and we have $n = 2^0 \cdot n$ is the form required.

Therefore, n is even, so there is some integer m such that $n = 2m$. Since n is a positive integer, this means $m < n$, and so m can be written in the form $m = 2^a b$ where a is nonnegative and b is odd. Then $n = 2^{a+1} b$. But this contradicts the assumption that n could not be written in this form!

Therefore, no such positive integer n exists, and so every integer can be written in this form. □

2. **Equivalence Relations**

(a) Let $\mathcal{R}$ be a relation on $\mathbb{Z}$ defined by $x\mathcal{R}y$ iff $x \equiv y \pmod 6$. Describe the equivalence classes for $\mathcal{R}$.

Solution: There are six equivalence classes:

$$[0] = \{6k : k \in \mathbb{Z}\}$$
$$[1] = \{6k + 1 : k \in \mathbb{Z}\}$$
$$[2] = \{6k + 2 : k \in \mathbb{Z}\}$$
$$[3] = \{6k + 3 : k \in \mathbb{Z}\}$$
$$[4] = \{6k + 4 : k \in \mathbb{Z}\}$$
$$[5] = \{6k + 5 : k \in \mathbb{Z}\}$$

(b) Let $\mathcal{R}$ be a relation on $\mathbb{Z}$ defined by $x\mathcal{R}y$ iff $x^3 + 3x \equiv y^3 + 3y \pmod 6$. Describe the equivalence classes for $\mathcal{R}$.

Solution: For an integer n, let us consider what are the possibilities for $n^3 + 3n$ mod 6.

$$n \equiv 0 \pmod 6 \implies n^3 + 3n \equiv 0 \pmod 6$$
$$n \equiv 1 \pmod 6 \implies n^3 + 3n \equiv 4 \pmod 6$$
$$n \equiv 2 \pmod 6 \implies n^3 + 3n = 14 \equiv 2 \pmod 6$$
$$n \equiv 3 \pmod 6 \implies n^3 + 3n = 36 \equiv 0 \pmod 6$$
$$n \equiv 4 \pmod 6 \implies n^3 + 3n = 76 \equiv 4 \pmod 6$$
$$n \equiv 5 \pmod 6 \implies n^3 + 3n = 140 \equiv 2 \pmod 6$$

Thus, the given relation has three equivalence classes:

$$\{6k : k \in \mathbb{Z}\} \cup \{6k+3 : k \in \mathbb{Z}\}$$
$$\{6k+1 : k \in \mathbb{Z}\} \cup \{6k+4 : k \in \mathbb{Z}\}$$
$$\{6k+2 : k \in \mathbb{Z}\} \cup \{6k+5 : k \in \mathbb{Z}\}$$

You could also write these as $\{3k : k \in \mathbb{Z}\}$, $\{3k+1 : k \in \mathbb{Z}\}$, $\{3k+2 : k \in \mathbb{Z}\}$.

(c) Let $\mathcal{R}$ be a relation on $\mathbb{Z}_6$ defined by $[x]\mathcal{R}[y]$ iff $([x] = [y]$ or $[x^2] = [y])$. List out the elements of $\mathcal{R}$ as ordered pairs $([x], [y])$. Is $\mathcal{R}$ an equivalence relation?

Solution: We have the elements $([0], [0]), ([1], [1]), ([2], [2]), ([3], [3]), ([4], [4]), ([5], [5])$ because by definition each $([n], [n])$ is in $\mathcal{R}$. We must also include pairs of the form $([n], [n^2])$. Trying all six possibilities for $[n]$ gives us

$$([0], [0]), ([1], [1]), ([2], [4]), ([3], [3]), ([4], [4]), ([5], [1])$$

So in total,

$$\mathcal{R} = \{([0], [0]), ([1], [1]), ([2], [2]), ([3], [3]), ([4], [4]), ([5], [5]), ([2], [4]), ([5], [1])\}$$

This is *not* an equivalence relation, as it is not symmetric! I.e. we have pairs $([x], [y])$ with the property that $[x^2] = [y]$ but $[y^2] \neq [x]$ - for example, $([2], [4])$ and $([5], [1])$.

(1) **Hints**
   (a) (Proofs a) Proof by induction. There's also a way without induction.
   (b) (Proofs b) If n is an odd number, then the next odd number is $n+2$, then $n+4$, then $n+6$, and so on. If n can be written in the form $3a + 5b + 7c$, can you prove that $n+6$ or $n+8$ can be written in this form? What about $n+4$ or $n+2$?
   (c) (Proofs c) Proof by contradiction.
   (d) (Proofs d) Proof by induction.
   (e) (Proofs e) Proof by induction, with base case $n = 3$.
   (f) (Proofs f) You can do this by strong induction, or by minimum counterexample.
   (g) (Equivalence Classes b) If you take an integer x, what are the possibilities for $x^3 + 3x$ mod 6?
   (h) (Equivalence Classes c) Remember that $\mathbb{Z}_6$ is just a six-element set. You can think of $\mathcal{R}$ as 'descended' from the relation on $\mathbb{Z}$ given by

$$x\mathcal{R}y \iff (x \equiv y \pmod 6) \vee (x^2 \equiv y \pmod 6)$$