MATH 220.201 CLASS 16 QUESTIONS

1. **Proofs**
   (a) Prove that $5 \mid 3^{4n+1} + 2$ for any nonnegative integer n.

   (b) Prove that there exists a positive integer N with the following property: every odd integer $n \geq N$ can be written in the form $n = 3a + 5b + 7c$ for some positive integers a, b, c.

   (c) Prove that if a and b are distinct natural numbers such that $\sqrt{a}$ and $\sqrt{b}$ are both irrational, then $\sqrt{a} + \sqrt{b}$ is also irrational.

   (d) Recall that the Fibonacci sequence is defined by $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that $2^n \geq F_{n+3}$ for $n \geq 3$.

   (e) Let $a_1 = 1, a_2 = 2$, and $a_n = \sum_{i=1}^{n-1} (i-1)a_i$ for $n \geq 3$. Prove that $a_n = (n-1)!$ for $n \geq 3$.

   (f) Prove that every positive integer can be written in the form $2^a b$ where a is a nonnegative integer and b is an odd integer.

2. **Equivalence Relations**
   (a) Let $\mathcal{R}$ be a relation on $\mathbb{Z}$ defined by $x\mathcal{R}y$ iff $x \equiv y \pmod 6$. Describe the equivalence classes for $\mathcal{R}$.
   (b) Let $\mathcal{R}$ be a relation on $\mathbb{Z}$ defined by $x\mathcal{R}y$ iff $x^3 + 3x \equiv y^3 + 3y \pmod 6$. Describe the equivalence classes for $\mathcal{R}$.
   (c) Let $\mathcal{R}$ be a relation on $\mathbb{Z}_6$ defined by $[x]\mathcal{R}[y]$ iff $([x] = [y]$ or $[x^2] = [y])$. List out the elements of $\mathcal{R}$ as ordered pairs $([x], [y])$. Is $\mathcal{R}$ an equivalence relation?

(1) **Hints**
   (a) (Proofs a) Proof by induction. There's also a way without induction.
   (b) (Proofs b) If n is an odd number, then the next odd number is $n + 2$, then $n + 4$, then $n + 6$, and so on. If n can be written in the form $3a + 5b + 7c$, can you prove that $n + 6$ or $n + 8$ can be written in this form? What about $n + 4$ or $n + 2$?
   (c) (Proofs c) Proof by contradiction.
   (d) (Proofs d) Proof by induction.
   (e) (Proofs e) Proof by induction, with base case $n = 3$.
   (f) (Proofs f) You can do this by strong induction, or by minimum counterexample.
   (g) (Equivalence Classes b) If you take an integer x, what are the possibilities for $x^3 + 3x$ mod 6?
   (h) (Equivalence Classes c) Remember that $\mathbb{Z}_6$ is just a six-element set. You can think of $\mathcal{R}$ as 'descended' from the relation on $\mathbb{Z}$ given by

$$x\mathcal{R}y \iff (x \equiv y \pmod 6) \vee (x^2 \equiv y \pmod 6)$$