MATH 220.201 CLASS 15 QUESTIONS

Remember: if A and B are sets, then a *relation* from A to B is a subset $R \subseteq A \times B$.

A *function* is a subset $f \subseteq A \times B$ such that for each $a \in A$, there is *exactly one* $b \in B$ such that $(a, b) \in f$. We also use the notation $f(a) = b$, and write $f : A \to B$ to say 'f is a function from A to B. The subset $f \subseteq A \times B$ is called the *graph* of the function.

A is called the *domain* of f.

B is called the *codomain* of f.

The *image* of a set $C \subseteq A$ (written $f(C)$) is the set $\{b \in B | \exists c \in C, f(c) = b\}$.

The *range* of f is $f(A)$.

The *preimage* of a set $D \subseteq B$ (written $f^{-1}(D)$) is the set $\{a \in A | f(a) \in D\}$.

(1) Prove that the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$ is injective, but not surjective.

Proof. First we prove that it is injective. Suppose that $x_1, x_2 \in [0, \infty)$ are such that $f(x_1) = f(x_2)$. Then $x_1^2 = x_2^2$, and so $(x_1 - x_2)(x_1 + x_2) = 0$. This means that either $x_1 = -x_2$ (impossible unless $x_1 = x_2 = 0$) or $x_1 = x_2$. Either way, $x_1 = x_2$, and so the function is injective.

To prove the function is not surjective, we see that any negative real number is not in the image. For example, there is no $x \in [0, \infty)$ such that $f(x) = -5$. □

(2) Prove that the function $f : \mathbb{R} \to [0, \infty)$ defined by $f(x) = x^2$ is surjective, but not injective.

Proof. First we prove that it is surjective. We must show that for any $y \in [0, \infty)$, there is some $x \in \mathbb{R}$ such that $x^2 = y$. Just pick $x = \sqrt{y}$.[1]

To prove the function is not injective, notice that $f(3) = f(-3) = 9$. These are two different inputs which give the same output, which means the function is not injective. □

(3) Prove that the function $f : \mathbb{R} \to (0, \infty)$ defined by $f(x) = 2^x$ is injective. (You can also prove that it is surjective without using logarithms, but it involves the Intermediate Value Theorem.)

Proof. Suppose that $x_1, x_2 \in \mathbb{R}$ are any numbers such that $f(x_1) = f(x_2)$. Then $2^{x_1} = 2^{x_2}$, and so $2^{x_1 - x_2} = 1$. But if $x_1 - x_2 > 0$, then $2^{x_1 - x_2} > 1$, and if

[1]This proof assumed that we already had the function $\sqrt{\cdot}$. To complete the proof without using this: we can find some small number a such that $a^2 < y$ and some very large number b such that $b^2 > y$. Since the function $f(x) = x^2$ is continuous, the Intermediate Value Theorem tells us there is some number x between a and b such that $x^2 = y$.

$x_1 - x_2 < 0$, then $2^{x_1 - x_2} < 1$. Therefore it must be that $x_1 - x_2 = 0$, and so $x_1 = x_2$. This proves that the function is injective. □

(4) Remember that for any positive integer n, $\mathbb{Z}_n = \{[0], [1], \ldots, [n-1]\}$ is the set of *equivalence classes* for the equivalence relation '$\equiv \pmod{n}$'.

(a) Prove that the function $f : \mathbb{Z} \to \mathbb{Z}_5$ defined by $f(x) = [x]$ is surjective, but not injective. (This isn't a trick question, it is just to check that you understand the definitions.

Proof. Pick any element of $\mathbb{Z}_5$ - i.e., an equivalence class of elements in $\mathbb{Z}$. It contains at least one element $x \in \mathbb{Z}$. Then this equivalence class is $[x]$. And so $f(x) = [x]$. Hence, the function f is surjective. It is not injective, though - for example, 1 and 6 both map to the class [1]. □

(b) Prove that the function $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ defined by $f([x]) = [3x+1]$ is a bijection.

Proof. First we'll prove that it's injective. Suppose that $[x]$ and $[y]$ are two equivalence classes (represented by integers x and y) such that $[3x+1] = [3y+1]$. This means that

$$3x + 1 \equiv 3y + 1 \pmod{5}$$

Then $5 \mid (3y+1) - (3x+1)$. This means $5 \mid 3(y-x)$. Since 5 is prime, and $5 \nmid 3$, this means $5 | y - x$, and so $x \equiv y \pmod{5}$. So $[x] = [y]$. Hence, the function is injective.

Now we prove it's surjective. Suppose that $[y]$ is an equivalence class, where $y \in \mathbb{Z}$.. We must show that $[y] = [3x+1]$ for some $x \in \mathbb{Z}$. If we let $x = 2y-2$, then

$$3x + 1 = 3(2y-2) + 1 = 6y - 5 \equiv 6y \equiv y \pmod{5}$$

and so we get $[y] = [3x+1]$. Therefore, the function is surjective. □