

MATH 220.201 CLASS 14 SOLUTIONS

(1) For each of the following relations f from A and B , is it a function? If it is, write an expression for $f(a)$ in terms of a . If not, explain why.

(a) $A = B = \{1, 2, 3, 4\}$, $f = \{(1, 2), (2, 3), (1, 3), (4, 4)\}$.

Solution: It is not a function, because $(1, 2)$ and $(1, 3)$ are both in f .

(b) $A = \mathbb{R} - \{1\}$, $B = \mathbb{R}$, $f = \{(a, b) \mid \frac{1}{a-1} = b\}$.

Solution: It is a function, and $f(a) = \frac{1}{a-1}$.

(c) $A = B = \mathbb{N}$, $f = \{(2n-1, n) \mid n \in \mathbb{N}\} \cup \{(2n, n) \mid n \in \mathbb{N}\}$.

Solution: It is a function, and

$$f(a) = \begin{cases} \frac{a}{2} & \text{if } a \text{ is even} \\ \frac{a+1}{2} & \text{if } a \text{ is odd} \end{cases}$$

This can also be written as $f(a) = \lceil a/2 \rceil$ (called the *ceiling* function of $a/2$).

(2) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = x^2$. Determine the following sets.

(a) $f([0, 4])$

(c) $f^{-1}([0, 9])$

Solution: $[0, 16]$

Solution: $[-3, 3]$

(b) $f([-1, 2])$

(d) $f^{-1}([1, 4])$

Solution: $[0, 4]$

Solution: $[0, 16]$

(3) Suppose that A, B are sets and $f : A \rightarrow B$ is a function.

(a) If $C \subseteq A$, is it necessarily true that $f^{-1}(f(C)) = C$?

Solution: No, not necessarily. Consider the function $f : \{1, 2\} \rightarrow \{3, 4\}$ defined by $f(1) = 4$ and $f(2) = 4$. Then let $C = \{1\}$. Then $f(C) = \{4\}$, and $f^{-1}(f(C)) = \{1, 2\}$. In general, $f^{-1}(f(C)) \supseteq C$.

(b) If $D \subseteq B$, is it necessarily true that $f(f^{-1}(D)) = D$?

Solution: No, not necessarily. Consider the same function as above and let $D = \{3\}$. Then $f^{-1}(D) = \emptyset$, and so $f(f^{-1}(D)) = \emptyset$. In general, $f(f^{-1}(D)) \subseteq D$.