MATH 220.201 CLASS 12 QUESTIONS

1. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = 2$, $a_2 = 1$, and

$$a_{n+1} = a_n + 6a_{n-1}$$

for $n \geq 2$. Prove that, for all n, $a_n = 3^{n-1} + (-2)^{n-1}$.

Proof. We prove this by strong induction on n.

- Base Cases: $n = 1, n = 2$.

$$3^{1-1} + (-2)^{1-1} = 1 + 1 = 2$$

$$3^{2-1} + (-2)^{2-1} = 3 - 2 = 1$$

- Inductive Step: suppose that $a_k = 3^{k-1} + (-2)^{k-1}$ for $k = 1, 2, \ldots, n$. We will show that $a_{n+1} = 3^n + (-2)^n$. We have that

$$\begin{aligned} a_{n+1} = a_n + 6a_{n-1} &= 3^{n-1} + (-2)^{n-1} + 6 \cdot 3^{n-2} + 6 \cdot (-2)^{n-2} \\ &= (3+6)3^{n-2} + (-2+6)(-2)^{n-2} \\ &= 9 \cdot 3^{n-2} + 4 \cdot (-2)^{n-2} = 3^n + (-2)^n \end{aligned}$$

This completes the induction.

□

2. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = 1$, $a_2 = 2$, and

$$a_{n+1} = 2a_n - a_{n-1} + 2$$

for all $n \geq 2$. Conjecture a formula for a_n and then prove your formula.

Proof. We will prove that $a_n = (n-1)^2 + 1$ for all $n \in \mathbb{N}$. We proceed by induction on n.

- Base cases: $n = 1, 2$. $1 = 0^2 + 1$ and $2 = 1^2 + 1$.
- Inductive step: suppose that $a_k = (k-1)^2 + 1$ for $k = 1, 2, \ldots, n$. We will show that $a_{n+1} = n^2 + 1$. We have that

$$\begin{aligned} a_{n+1} = 2a_n - a_{n-1} + 2 &= 2(n-1)^2 + 2 - (n-2)^2 - 1 + 3 \\ &= 2n^2 - 4n + 2 - (n^2 - 4n + 4) + 3 \\ &= n^2 + 1 \end{aligned}$$

This completes the induction.

□

3. For any positive integer, n is called *prime* if $n \geq 2$ and there exist no integers a such that $1 < a < n$ and $a|n$. Prime numbers are usually denoted by the letter p.

   Prove that any integer $n \geq 2$ is either prime or can be written as a product of (not necessarily distinct) primes.

   Proof. Suppose, for a contradiction, that there is some integer greater than or equal to 2 which is neither prime nor a product of primes. Let n be the least such integer. Then because n is not prime, there exists some $a \in \mathbb{Z}$ such that $1 < a < n$ and $a|n$. So there is some $b \in \mathbb{Z}$ such that $n = ab$. Because $a > 1$, $b < n$, and because $a < n$, $b > 1$. By assumption, a is either prime or is a product of primes. Similarly, b is either prime or is a product of primes. Therefore, n is a product of primes. This contradicts our assumption about n.

   Therefore, no such n exists, which completes the proof. □

4. (Binary Representation) Prove that any positive integer n can be written as

$$n = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k}$$

   for some integers $i_1, \ldots, i_k$ with the property that $0 \leq i_1 < i_2 < \cdots < i_k$. (You may assume the fact that for any positive integer n, there is a unique greatest integer i such that $2^i \leq n$.)

   Can you prove this representation is *unique*?

   Proof. Call a sum of powers of 2 as shown above *binary representation.* We prove by strong induction on $n$that every positive integer n has a binary representation.
   - Base case: $n = 1$. $1 = 2^0$.
   - Inductive step: Let n be an integer greater than 1, and suppose that $1, 2, \ldots, n-1$ all have binary representations. Let i be the greatest integer with the property that $2^i \leq n$. Then, $n - 2^i \geq 0$. If $n - 2^i = 0$, then we are done as $n = 2^i$. If not, then $1 \leq n - 2^i \leq n - 1$, and therefore $n - 2^i$ can be written in the form

$$n - 2^i = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_k}$$

     where $i_1, i_2, \ldots, i_k$ are integers such that $0 \leq i_1 < i_2 < \cdots < i_k$. We have that $i_k < i$, because if $i_k \geq i$, we would have $n - 2^i \geq 2^{i_k} \geq 2^i \implies n \geq 2^{i+1}$ which contradicts the maximality of i. Therefore,

$$n = 2^{i_1} + 2^{i_2} + \cdots + 2^{i_k} + 2^i$$

     is a binary representation of n. This completes the proof.

□

   Proof that binary representation is unique. Suppose that some positive integer has two binary representations. Let n be the smallest such positive integer, and suppose it has two nonidentical binary representations

$$2^{i_1} + 2^{i_2} + \ldots + 2^{i_k} = n = 2^{j_1} + 2^{j_2} + \ldots + 2^{j_\ell}$$

If $i_1 = j_1$, then we have two binary representations

$$2^{i_2} + \ldots + 2^{i_k} = n - 2^{i_1} = n - 2^{j_1} = 2^{j_2} + \ldots + 2^{j_\ell}$$

By the assumption about the minimality of n, the two representations must be the same, and therefore, our two binary representations of n are the same, which is a contradiction. Hence, assume that $i_1 \neq j_1$. WLOG, $i_1 < j_1$. Then

$$2^{i_1} + 2^{i_2} + \ldots + 2^{i_k} \equiv 2^{i_1} \pmod{2^{i_1+1}}$$

$$2^{j_1} + 2^{j_2} + \ldots + 2^{j_\ell} \equiv 0 \pmod{2^{i_1+1}}$$

which is a contradiction, because the two sums are equal. Thus, no such n exists, which proves the result. □