MATH 220.201 CLASS 11 SOLUTIONS

1. Prove that if $n \geq 2$ is a natural number and $A_1, A_2, \ldots, A_n$ are sets, then

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$$

Proof. We use induction on n.

- Base case: $n = 2$. This is just De Morgan's Law for sets.
- Inductive step: $P(n) \implies P(n+1)$. Suppose that, for any n sets, the complement of the union is the intersection of their complements. Now suppose we have $n+1$ sets $A_1, A_2, \ldots, A_{n+1}$. Then

$$\begin{aligned}
\overline{A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}} &= \overline{(A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1}} \\
&= \overline{(A_1 \cup A_2 \cup \cdots \cup A_n)} \cap \overline{A_{n+1}} \\
&= (\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}) \cap \overline{A_{n+1}} \\
&= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} \cap \overline{A_{n+1}}
\end{aligned}$$

The second step holds by $P(2)$ and the third step holds by $P(n)$.

□

2. Prove that for every integer $n \geq 5$, $2^n > n^2$.

Proof. We use induction on n.

- Base case: $n = 5$. This is the statement that $2^5 > 5^2$, i.e. $32 > 25$.
- Inductive step: $P(n) \implies P(n+1)$. Suppose that $2^n > n^2$. We want to show that $2^{n+1} > (n+1)^2$. It is sufficient for us to show that $2^{n+1} - 2^n > (n+1)^2 - n^2$, i.e. to show that $2^n > 2n + 1$. This is true because

$$2^n > n^2 > 3n > 2n + 1$$

The first inequality is by the inductive hypothesis, and the second inequality is because $n > 3$, and the third inequality because $n > 1$. Therefore, $2^{n+1} > (n+1)^2$ and this completes the induction.

□

Note: In the inductive step, we showed that the left side increases more than the right side. We could have instead proved that $\frac{2^{n+1}}{2^n} > \frac{(n+1)^2}{n^2}$. Simplifying both sides, this becomes $2 > \left(1 + \frac{1}{n}\right)^2$ - this is true for $n = 5$, and it therefore is true for all larger n as well.

3. Prove that for every positive odd integer n, $5 | 4^n + 1$.

Proof. Any odd integer n can be written in the form $n = 2k + 1$ for some integer k. If n is positive, then this means $k \geq 0$. So it is equivalent to prove that for every integer $k \geq 0$, $5 | 4^{2k+1} + 1$. We prove this by induction on k.

- Base Case: $k = 0$. $4^1 + 1 = 5$ so the divisibility clearly holds.
- Inductive step: Suppose that $5|4^{2k+1}+1$. We wish to show that $5|4^{2(k+1)+1}+1$, that is, $5|4^{2k+3} + 1$. By the inductive hypothesis, $4^{2k+1} \equiv -1 \pmod 5$, and therefore

$$4^{2k+3} = 4^{2k+1} \cdot 4^2 \equiv (-1) \cdot 1 \pmod 5 \equiv -1 \pmod 5$$

  and therefore, $5|4^{2k+3} + 1$, which completes the induction.

□

4. Prove that every natural number $n \geq 8$ can be written in the form $n = 5a + 3b$ where a, b are nonnegative integers.

Proof 1. Let the described property be denoted by $P(n)$. We will show that $P(8), P(9)$, and $P(10)$ are all true, and then prove that $P(n) \implies P(n + 3)$ for all $n \geq 8$. This will separately show that $P(8 + 3k), P(9 + 3k)$, and $P(10 + 3k)$ hold for all nonnegative integers k, thereby proving $P(n)$ for all $n \geq 8$.

- Base case: $n = 8, 9, 10$. $8 = 5(1) + 3(1)$, $9 = 5(0) + 3(3)$, $10 = 5(2) + 3(0)$.
- Inductive step: suppose that $n = 5a + 3b$ for some nonnegative integers a, b. Then $n+3 = 5a+3(b+1)$. $b+1$ is also a nonnegative integer, so this implies $P(n + 3)$. This completes the induction.

□

Proof 2. Let $P(n)$ denote the property described above, and let $Q(n) = P(n) \wedge P(n + 1) \wedge P(n + 2)$. That is, $Q(n)$ is the property that $n, n + 1, n + 2$ can all be written in the given form. We will prove $\forall n \geq 8, Q(n)$ by induction on n.

- Base case: $n = 8$. $8 = 5(1) + 3(1)$, $9 = 5(0) + 3(3)$, $10 = 5(2) + 3(0)$.
- Inductive step: Suppose that $Q(n)$ is true: namely, that $n = 5a+3b$, $n+1 = 5c+3d$, $n+2 = 5e+3f$. We must show that $n+1, n+2$, and $n+3$ have the required property P. By assumption, $n + 1 = 5c + 3d$, $n + 2 = 5e + 3f$, and $n+3 = 5a+3(b+1)$. Thus, $Q(n+1)$ is true, which completes the induction.

□

5. Consider the following statement.

For every integer $k \geq 5$, there exists a natural number N such that for every integer $n \geq N$, $2^n > n^k$.

(a) What is wrong with the following argument disproving the statement?

Proof. Suppose there is such an N. Then for every $k \geq 5$ and every $n \geq N$, we have $2^n > n^k$. This means $\log_n(2^n) > k$. But this doesn't hold when $k \geq \log_n(2^n)$. This is a contradiction! □

Solution: The statement is saying that N can be chosen dependent on k, whereas the proof is assuming that N is chosen *independent* of k. The reason N is dependent on k, is that the part quantifying N is inside the sentence quantified by k.

$$\forall k \geq 5, (\exists N \in \mathbb{N}, (\forall n \geq N, (2^n > n^k)))$$

(b) I think there is a better way to name the variables:

For every integer $k \geq 5$, there exists a natural number N_k such that for every integer $n \geq N_k$, $2^n > n^k$.

Why is this better? **This reminds us that the value of N that is chosen is dependent on what k is.**

(c) Can you prove the statement? (This is challenging and may require multiple steps! Hint: base case $N_k = 2^k$.)

Proof. For each $k \geq 5$, let $N_k = 2^k$. We prove that, for all $n \geq 2^k$, $2^n > n^k$ by induction on n.

- Base case: $n = 2^k$. We must check that, for each $k \geq 5$, $2^{(2^k)} > (2^k)^k$. But $(2^k)^k = 2^{(k^2)}$. The inequality $2^{(2^k)} > 2^{(k^2)}$ for $k \geq 5$ now follows because we proved in Q2 that $2^k > k^2$ for $k \geq 5$.
- Inductive step: Suppose that $2^n > n^k$. We must show that $2^{n+1} > (n+1)^k$. It is sufficient to show that $\frac{2^{n+1}}{2^n} > \frac{(n+1)^k}{n^k}$, as then we have that

$$2^{n+1} = 2^n \cdot \frac{2^{n+1}}{2^n} > 2^n \cdot \frac{(n+1)^k}{n^k} = \frac{2^n}{n^k} \cdot (n+1)^k > (n+1)^k$$

So let's show that $\frac{2^{n+1}}{2^n} > \frac{(n+1)^k}{n^k}$. Simplifying both sides, it is equivalent to show that $2 > \left(1 + \frac{1}{n}\right)^k$. Since $n \geq 2^k$, it is sufficient to show that $2 > \left(1 + \frac{1}{2^k}\right)^k$. When we expand the binomial product on the right hand side, we get 2^k terms: one of them is equal to 1, and every other term is at most $\frac{1}{2^k}$. Therefore, the right hand side is at most $1 + \frac{2^k - 1}{2^k} = 2 - \frac{1}{2^k}$. Therefore, the inequality $2 > \left(1 + \frac{1}{2^k}\right)^k$ holds, and this proves that the induction holds.

□