MATH 220.201 CLASS 10 QUESTIONS

Use induction to prove the following results.

1. For all $n \in \mathbb{N}$, $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

   Note: as this is the first induction proof, I will make it extra-explicit.

   Proof. Let $P(n)$ be the sentence $(1 + 2 + \ldots + n = \frac{n(n+1)}{2})$. We wish to prove $\forall n \in \mathbb{N}, P(n)$. We prove this by induction on n.
   - **Base Case:** $P(1)$**.** This is equivalent to showing that $1 = \frac{1 \cdot 2}{2}$. This is clear.
   - **Inductive Step:** $\forall n \in \mathbb{N}, P(n) \implies P(n+1)$**.** We must prove that for any n, if $P(n)$ is true, then $P(n+1)$ is true. Suppose that $P(n)$ is true, i.e. $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.[1] Then

$$\begin{aligned}
1 + 2 + \ldots + n + (n+1) &= 1 + 2 + \ldots + n + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \qquad \text{by the inductive hypothesis} \\
&= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}$$

   Therefore, we have proven $P(n+1)$. This completes the induction, and therefore completes the proof.

   □

2. For all $n \in \mathbb{N}$, $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

   Proof. We prove it by induction on n. Let $P(n)$ be the sentence $(1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6})$.
   - **Base Case:** $P(1)$**.** We must check that $1^2 = \frac{1 \cdot 2 \cdot 3}{6}$. This is clear.

[1] This is called the *inductive hypothesis*.

- **Inductive Step:** $P(n) \implies P(n+1)$**.** Suppose that $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ is true. Then

$$\begin{aligned}
1^2 + 2^2 + \ldots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \qquad (by inductive hypothesis) \\
&= \frac{(n+1)(2n^2+n)}{6} + \frac{(n+1)(6n+6)}{6} \\
&= \frac{(n+1)(2n^2+7n+6)}{6} \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}
\end{aligned}$$

which proves $P(n+1)$. This completes the induction.

□

3. Let $x > -1$ be a real number. Then for all $n \in \mathbb{N}, (1+x)^n \geq 1 + nx$.

Proof. We use a proof by induction on n.[2] Let $P(n)$ be the sentence

$$P(n) := \forall x > -1, ((1+x)^n \geq 1 + nx)$$

- **Base Case:** $P(1)$ is the statement that, for all x, $(1+x)^1 \geq 1 + 1 \cdot x$. The two sides are equal, so the weak inequality $\geq$ holds.
- **Inductive Step:** Suppose that $\forall x > -1, ((1+x)^n \geq 1 + nx)$. We wish to prove that $\forall x > -1, ((1+x)^{n+1} \geq 1 + (n+1)x)$. Let x be an arbitrary real number greater than -1. Then

$$\begin{aligned}
(1+x)^{n+1} &= (1+x)^n(1+x) \\
&\geq (1+nx)(1+x) \qquad \text{(by inductive hypothesis and because } 1+x > 0) \\
&= 1 + nx + x + nx^2 \\
&= 1 + (n+1)x + nx^2 \\
&\geq 1 + (n+1)x \qquad \text{because } nx^2 \geq 0
\end{aligned}$$

We have thus proven $P(n+1)$ and this completes the induction.

□

4. Let A be a finite set of size n. Then $|\mathcal{P}(A)| = 2^n$.

Proof. Let $P(n)$ be the sentence

$$P(n) := \text{ For any set } A \text{ of size } n, |\mathcal{P}(A)| = 2^n$$

[2] You cannot induct on x, because x is taken to be a *real* number! So you have to keep x arbitrary throughout this proof - no substituting of $x+1$ for x or anything like that. Make sure you know what your statement $P(n)$ is before you start the induction.

We prove $\forall n \in \mathbb{N}, P(n)$ by induction on n.

- **Base Case:** Suppose that A is a set of size 1. Then it has a single element, x. Then A has two subsets: $\emptyset$ and $\{x\}$. So $|\mathcal{P}(A)| = 2 = 2^1$, which proves $P(1)$.
- **Inductive step:** Assume $P(n)$ is true. Now suppose that A is an arbitrary set of size $n+1$. Call its elements $x_1, x_2, \ldots, x_{n+1}$. For any subset $S \subseteq A$, exactly one of $x_{n+1} \notin S$ or $x_{n+1} \in S$ holds true.
  (a) In the first case, $S \subseteq \{x_1, \ldots, x_n\}$. By the inductive hypothesis, there are exactly 2^n such subsets S.
  (b) In the second case, $S - \{x_{n+1}\} \subseteq \{x_1, \ldots, x_n\}$. By the inductive hypothesis, there are exactly 2^n such subsets $S - \{x_{n+1}\}$.

  Since the two cases considered are mutually exclusive, the total number of subsets $S \subseteq A$ is $2^n + 2^n = 2^{n+1}$. Thus $|\mathcal{P}(A)| = 2^{n+1}$, which completes the induction.

$\square$

(1) For all $n \in \mathbb{N}$, $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$. Then use this to prove that

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots = 1$$

Proof. Let $P(n)$ be the sentence $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$. We prove this by induction on n.

- **Base Case:** For $n = 1$, the sentence is $\frac{1}{1\cdot 2} = \frac{1}{1+1}$. This is obvious.
- **Inductive Step:** Suppose that $P(n)$ holds true. Then

$$\begin{aligned}
\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{n+1}{n+2}
\end{aligned}$$

We have proven $P(n+1)$, and so this completes the induction.

Now we have proven that $P(n)$ holds true for all n. The infinite sum $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \ldots$ can be written as $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$. And therefore

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

□

6. For all $n \in \mathbb{N}$, $\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}$ is an integer. Then show that for all $n \in \mathbb{N}$, $\frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}$ is an integer.

Note: The numbers $\frac{n(n+1)}{2}$ are called *triangular numbers* because that number of dots can be arranged into a triangle. The numbers $\frac{n(n+1)(n+2)}{6}$ are called *tetrahedral numbers* for a similar reason.

Proof of first assertion. We prove it by induction on n.

- **Base Case:** $n = 1$. This is equivalent to showing that $\frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3}$ is an integer. This is obvious.
- **Inductive Step:** Suppose that $\frac{n(n+1)(n+2)}{6}$ is an integer. We wish to show that $\frac{(n+1)(n+2)(n+3)}{6}$ is an integer. But

$$\frac{(n+1)(n+2)(n+3)}{6} = \frac{n(n+1)(n+2)}{6} + \frac{3(n+1)(n+2)}{6} = \frac{n(n+1)(n+2)}{6} + \frac{(n+1)(n+2)}{2}$$

and because of Question 1, $\frac{(n+1)(n+2)}{2}$ is an integer. Therefore, $\frac{(n+1)(n+2)(n+3)}{6}$ is an integer, which completes the induction.

□

Proof of section assertion. Again we prove it by induction on n.

- **Base Case:** $n = 1$. This is equivalent to showing that $\frac{1\cdot 2\cdot 3\cdot 4}{1\cdot 2\cdot 3\cdot 4}$ is an integer. This is obvious.
- **Inductive Step:** Suppose that $\frac{n(n+1)(n+2)(n+3)}{24}$ is an integer. We wish to show that $\frac{(n+1)(n+2)(n+3)(n+4)}{4}$ is an integer. But

$$\begin{aligned}\frac{(n+1)(n+2)(n+3)(n+4)}{24} &= \frac{n(n+1)(n+2)(n+3)}{24} + \frac{4(n+1)(n+2)(n+3)}{24} \\ &= \frac{n(n+1)(n+2)(n+3)}{24} + \frac{(n+1)(n+2)(n+3)}{6}\end{aligned}$$

and because of the previous assertion, $\frac{(n+1)(n+2)(n+3)}{6}$ is an integer. Therefore, $\frac{(n+1)(n+2)(n+3)(n+4)}{24}$ is an integer, which completes the induction.

□

Challenge: Can you form a more general statement and prove it by induction?

7. For all $n \in \mathbb{N}$, $3 \mid 2^n + 1 \iff 3 \nmid 2^n - 1$.

Proof. Let

$$Q(n) := (3 \mid 2^n + 1) \wedge (3 \nmid 2^n - 1)$$
$$R(n) := (3 \nmid 2^n + 1) \wedge (3 \mid 2^n - 1)$$

We wish to show $\forall n \in \mathbb{N}, (Q(n) \vee R(n))$. We prove this by induction on n.

- **Base Case:** When $n = 1$, $2^n + 1 = 0$ and $2^n - 1 = 2$. Therefore, $R(1)$ is true and $Q(n)$ is false. Thus, $Q(1) \vee R(1)$ is true.

- **Inductive Step:** We wish to prove $(Q(n) \vee R(n)) \implies (Q(n+1) \vee R(n+1))$. This is logically equivalent to showing $Q(n) \implies (Q(n+1) \vee R(n+1))$ and $R(n) \implies (Q(n+1) \vee R(n+1))$. We will show that $Q(n) \implies R(n+1)$ and $R(n) \implies Q(n+1)$, which is sufficient.
  (a) Suppose $Q(n)$. Then $3 \mid 2^n + 1$, so $2^n \equiv -1 \pmod 3$. Then
  $$2^{n+1} \equiv -2 \equiv 1 \pmod 3$$
  and so $3 \mid 2^n - 1$ and $3 \nmid 2^n + 1$. We therefore have $R(n+1)$.
  (b) Suppose $R(n)$. Then $3 \mid 2^n - 1$, so $2^n \equiv 1 \pmod 3$. Then
  $$2^{n+1} \equiv 2 \equiv -1 \pmod 3$$
  and so $3 \mid 2^n + 1$ and $3 \nmid 2^n - 1$. We therefore have $Q(n+1)$.

□

Note: Here's a diagram of the logical flow of the last proof.