MATH 220 CLASS 1: A SIMPLE PROOF

This document is meant to serve two purposes. First, it includes the short proof I presented in class today (January 4). Second, it serves as a template file for you to learn some basic LaTeX typesetting. I have tried to include a variety of LaTeX commands.

1. A theorem about divisibility

Here's a stronger statement than the one I proved in class.

Proposition 1.1. *Let n be an odd integer. Then $n^2 - 1$ is a multiple of 8.*

Note: A priori, it is not even clear why $n^2 - 1$ should be divisible by 4. So the above fact is quite surprising!

Proof. The set of odd integers can be described as

$$\{\text{Odd integers}\} = \{2k+1 | k \in \mathbb{Z}\}$$

Therefore, there is some integer k such that $n = 2k + 1$. So one can write $n^2 - 1$ as

$$\begin{aligned} n^2 - 1 &= (2k+1)^2 - 1 \\ &= 4k^2 + 4k + 1 - 1 \\ &= 4k(k+1) \end{aligned} \tag{1.1}$$

Because k and $k + 1$ are consecutive integers, one of them is even, and therefore is divisible by 2. Therefore, $4k(k + 1)$ is divisible by 8, as desired. □

Here is an alternate proof. I have written the proof more concisely, and moved the very rigorous step to a footnote.

Proof. $n^2 - 1$ can be factored as

$$n^2 - 1 = (n-1)(n+1)$$

Since n is odd, $(n-1)$ and $(n+1)$ must each be even. Moreover, since they are *consecutive* even numbers, one of them is a multiple of 4.[1] Therefore, $(n - 1)(n + 1)$ is a multiple of $2 \cdot 4 = 8$. □

[1] This can be seen by letting $n - 1 = 2k$ and $n + 1 = 2k + 2$. Then if k is even, $n - 1$ is a multiple of 4, while if k is odd, $n + 1$ is a multiple of 4.

2. Is $\sqrt{2} \in \mathbb{Q}$?

One of the other mysterious questions I posed in class today was the following.

Question 2.0.1. *Can $\sqrt{2}$ be expressed in the form $\frac{a}{b}$, where $a, b \in \mathbb{Z}$? That is, is $\sqrt{2}$ rational?*

The answer, as it turns out, is no. Apparently this was a big deal to some Greek mathematicians, who had the implicit assumption that all the numbers and quantities they normally would encounter would be rational numbers. But $\sqrt{2}$ appears as the hypotenuse of a right triangle with both legs of length 1. The proof is below, but the reasoning is a bit more advanced. Proceed at your own peril!

Proof. We will argue by contradiction. That is, we will assume that $\sqrt{2}$ can be written in the form $\frac{a}{b}$ for some integers a and b, and then show that we can reach an impossible conclusion, thereby invalidating this initial assumption.

Suppose, for a contradiction, that $\sqrt{2} = \frac{a}{b}$ for some integers a and b. Without loss of generality, we may assume that this fraction is *reduced* - that a and b have no common factors. This is true because any fraction can be written in reduced form. Then,

$$\sqrt{2} = \frac{a}{b} \implies 2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$
$$\implies 2b^2 = a^2$$

The left side of this equation is even, and therefore, a is even. Write $a = 2x$ for some integer x.

$$2b^2 = (2x^2) = 4x^2$$
$$b^2 = 2x^2$$

The right side of this equation is even, and therefore, b is even. This means that a and b share a common factor of 2. But we assumed that a and b have no common factors! So we have reached a contradiction.

Therefore, our assumption that such a fraction $\frac{a}{b}$ existed, was incorrect. Therefore, $\sqrt{2}$ is irrational. $\square$