MATH 21B, FEBRUARY 23: ORTHONORMAL BASES AND PROJECTION

Definition: Let $\vec{v}, \vec{w} \in \mathbb{R}^n$. Then we say $\vec{v}$ and $\vec{w}$ are *orthogonal* if the *dot product*

$$\vec{v} \cdot \vec{w} = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1 w_1 + \ldots + v_n w_n$$

is equal to 0. In general, the dot product of $\vec{v}$ and $\vec{w}$ is equal to $|\vec{v}||\vec{w}|\cos\theta$ where θ is the angle between $\vec{v}$ and $\vec{w}$. The number $\cos\theta$ is called the *correlation coefficient*: when it is positive, the vectors are *positively correlated*, when it is negative, the vectors are *negatively correlated*, and when it is zero, the vectors are orthogonal. Note that $\vec{v} \cdot \vec{v} = |\vec{v}|^2$ has correlation coefficient 1.

(1) Suppose that $\vec{u}_1, \ldots, \vec{u}_m$ is a set of vectors in $\mathbb{R}^n$ such that each is orthogonal to all of the others. Must they be linearly independent? How do you know?

Definition: A set of vectors $\vec{u}_1, \ldots, \vec{u}_m$ in V is called an *orthonormal basis for V* if they form a basis for V, each has length 1, and each is orthogonal to all of the others. Expressing this condition another way,

$$\vec{u}_i \cdot \vec{u}_j = \begin{cases} 1 & \text{if } \ i = j \\ 0 & \text{if } \ i \neq j \end{cases}$$

(2) Let V be a subspace of $\mathbb{R}^n$. Any vector $\vec{x}$ in $\mathbb{R}^n$ can be written as $\vec{x} = \mathrm{proj}_V(\vec{x}) + \vec{x}^\perp$, where $\mathrm{proj}_V(\vec{x})$ is the *orthogonal projection of $\vec{x}$ onto V*, and $\vec{x}^\perp$ is orthogonal to V. Suppose that we have an orthonormal basis $(\vec{u}_1, \ldots, \vec{u}_m)$ for V.

(a) Explain why $\mathrm{proj}_V(\vec{x})$ can be written as $c_1\vec{u}_1 + \ldots + c_m\vec{u}_m$ for some scalars $c_1, \ldots, c_m$.

(b) How do you calculate these coefficients c_i? (Hint: Take the equation $\vec{x} = (c_1\vec{u}_1 + \ldots + c_m\vec{u}_m) + \vec{x}^{\perp}$ and take the dot product of both sides with $\vec{u}_i$'s.)

(c) Use this to write a formula for $\mathrm{proj}_V(\vec{x})$ in terms of $\vec{x}$ and our basis $\vec{u}_1, \ldots, \vec{u}_m$. At what step did you require the basis to be *orthonormal*?

(3) Let V be the plane $2x + 2y + z = 0$, $\vec{u}_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$, and $\vec{u}_2 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$. Let $\vec{x} = \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix}$.

(a) Verify that $(\vec{u}_1, \vec{u}_2)$ is an orthonormal basis of V.

(b) Find $\mathrm{proj}_V(\vec{x})$. (Check that your answer is reasonable by computing the difference $\vec{x} - \mathrm{proj}_V(\vec{x})$. What should be true about this vector?)

(4) Let V be an m-dimensional subspace of $\mathbb{R}^n$. Consider the linear transformation $\text{proj}_V : \mathbb{R}^n \to \mathbb{R}^n$.

(a) What is $\text{im}(\text{proj}_V)$? What is its dimension?

(b) What is $\ker(\text{proj}_V)$? What is its dimension?

(c) If you have a basis $\vec{u}_1, \ldots, \vec{u}_m$ for V, how would you calculate $\ker(\text{proj}_V)$?

(5) Suppose $\mathfrak{B} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is an orthonormal basis of $\mathbb{R}^3$. In each part of this problem, you are given the $\mathfrak{B}$-matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$. Describe the linear transformation geometrically.

(a) $\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

If $\mathfrak{B}$ were *not* orthonormal, how would your answers change?

(6) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 3 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}$; these three vectors are linearly independent. Let V be the subspace of $\mathbb{R}^4$ spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Find an orthonormal basis of V.