MATH 21B, FEBRUARY 21: LINEAR SPACES

Definition: A *linear space* is a set X with a commutative *addition* operation, $+$, satisfying three properties

(1) (Closed under addition) If x and y are elements of X, then $x + y$ is in X.
(2) (Closed under scalar multiplication) If x is in X and λ is a real number, then λx is in X.
(3) (Existence of an additive identity) There is an element $\mathbf{0}$ in X such that, for any x in X, $\mathbf{0} + x = x = x + \mathbf{0}$.

For example, any linear subspace of $\mathbb{R}^n$ (including the zero subspace, and $\mathbb{R}^n$ itself) is itself a linear space.

(1) Which of the following are linear spaces?

(a) The set of all 2×2 matrices.
Answer: Yes

(b) The set of all polynomials of degree 3 or less. (i.e., functions of the form $c_3x^3 + c_2x^2 + c_1x + c_0$)
Answer: Yes

(c) The set of all polynomials of degree 3 or less such that $P(5) = 0$.
Answer: Yes

(d) The set of all 2×2 matrices with lower left entry equal to 0.
Answer: Yes

(e) The set of all polynomials of degree 3 or less such that $P(5) = 3$.
Answer: No

(f) The set of all real-valued functions on the interval $[1, 3]$.
Answer: Yes

(g) The set of all *continuous* real-valued functions on the interval $[1, 3]$.
Answer: Yes

(h) The set of all *positive* real-valued functions on the interval $[1, 3]$.
Answer: No

(2) For each of the following linear spaces, give the dimension and find a basis (or say in a few words why it is impossible).

(a) The space $\mathcal{P}_3$ of all polynomials of degree 3 or less.
Solution: $\{1, x, x^2, x^3\}$ is a basis for this space, because any polynomial of degree 3 or less can be expressed as a linear combination of these four monomials, and these four monomials are linearly independent (i.e., no nontrivial linear combination yields the zero polynomial). Therefore, $\mathcal{P}_3$ has dimension 4. (Question: in general, if we replaced 'degree 3' by 'degree n' for some number n, what would the dimension be? What if you let $n \to \infty$, i.e., considering the space of *all* polynomials?)

(b) The subspace V of $\mathcal{P}_3$ consisting of all polynomials of degree 3 or less such that $P(5) = 0$.
Solution 1: Any polynomial P such that $P(5) = 0$ (i.e. with 5 as a root) must be divisible by $x - 5$. Therefore, $P(x) = (x-5)Q(x)$ for some polynomial Q. Since P has degree 3 or less, Q has degree 2 or less, i.e. $Q(x) = c_2x^2 + c_1x + c_0$ for some constants c_2, c_1, c_0. So V has a basis given by the polynomials $\{x-5, x^2-5x, x^3-5x^2\}$, and has dimension 3.

Solution 2: Any polynomial P of degree 3 or less can be written in the form

$$P(x) = a_3(x-5)^3 + a_2(x-5)^2 + a_1(x-5) + a_0$$

for some constants a_3, a_2, a_1, a_0. That is, $\{(x-5)^3, (x-5)^2, (x-5), 1\}$ is a perfectly good basis for $\mathcal{P}_3$. The subspace V of polynomials P such that $P(5) = 0$, is the span of the first three of these basis vectors (and therefore has dimension 3).

(c) The space $\mathcal{C}_{[1,3]}$ of continuous functions on $[1, 3]$.
Solution: This is an *infinite dimensional* space! Here's a short explanation of why (the following is not something you need to know now, but these ideas will appear towards the end of the course and I think it's an interesting extension of what we have been doing). Remember that we say a space has *dimension* n if it has a basis of size n: i.e., a set of elements $\{v_1, \ldots, v_n\}$ which are linearly independent and span the space. So we'll show that $\mathcal{C}_{[1,3]}$ contains an *infinite* sequence of elements $\{f_1, f_2, f_3, \ldots\}$ which are linearly independent. There are lots and lots of such infinite sequences, here are two examples:

$$\{1, x, x^2, x^3, x^4, x^5, \ldots\}$$
$$\{\sin(x), \sin(2x), \sin(3x), \sin(4x), \sin(5x), \sin(6x)\}$$

The first sequence, for example, form a linearly independent set, because if there were some linear relation between the powers of x, that would mean we have a nontrivial polynomial $P(x)$ (i.e. not the zero polynomial) which is equal to zero on the entire interval $[1, 3]$. That would mean every number in $[1, 3]$ is a root of $P(x)$, and therefore $P(x)$ has infinitely many roots! That's not possible for a finite-degree polynomial! Therefore, we can't have such a nontrivial relation P. (From the point of view of Math 1b - if $P(x)$ is equal to zero on the interval $[1, 3]$, then let's Taylor approximate it

around $x = 2$: all of its derivatives at $x = 2$ are zero, and therefore its Taylor series would just be 0.)

(d) The space of 2×3 matrices A such that $A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Solution: This is a subspace of the 6-dimensional space of 2×3 matrices. Let's suppose that A is some matrix such that $A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and let's see what we can figure out about A. Write

$$A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$$

First, the equation $A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is equivalent to the following two equations:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = [0] \qquad \begin{bmatrix} x_{21} & x_{22} & x_{23} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = [0]$$

That is, the two vectors $\begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}$ and $\begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}$ are both *orthogonal* to $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. In other words, these two vectors (which are the two rows of A) lie in *the plane orthogonal to* $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

We know how to find a basis for this plane, which is described by the equation $x + y + 2z = 0$. It's done by finding the kernel of the matrix $\begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$. You've done this on your homework, so I'll just say what the basis is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

So each of $\begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \end{bmatrix}$ and $\begin{bmatrix} x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}$ is a linear combination of these two vectors. Therefore, the space of 2×3 matrices A such that $A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has basis

$$\left\{ \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \right\}$$

that is, it is 4-dimensional.

(e) The space of 2×3 matrices A such that $A \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution: Again, let $A = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix}$, then the given matrix equation is equivalent to the four equations

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [0] \qquad \begin{bmatrix} x_{11} & x_{12} & x_{13} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = [0]$$

$$\begin{bmatrix} x_{21} & x_{22} & x_{23} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = [0] \qquad \begin{bmatrix} x_{21} & x_{22} & x_{23} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = [0]$$

In other words, each row of A is orthogonal to both $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$. The set of vectors orthogonal to both of these, is just the kernel of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}$. I think this was also on the last homework: the kernel is spanned by $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$. Therefore, the space of 2×3 matrices A such that $A \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ has basis

$$\left\{ \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix} \right\}$$

(3) Find a matrix A such that the image of the matrix $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$ coincides with the kernel of the matrix A.

Solution: We want a matrix whose kernel is $\left\langle \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\rangle$. As we saw in the previous part, $A = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$ fits the bill.