MATH 21B, FEBRUARY 21: LINEAR SPACES

Definition: A *linear space* is a set X with a commutative *addition* operation, $+$, satisfying three properties

(1) (Closed under addition) If x and y are elements of X, then $x + y$ is in X.
(2) (Closed under scalar multiplication) If x is in X and λ is a real number, then λx is in X.
(3) (Existence of an additive identity) There is an element $\mathbf{0}$ in X such that, for any x in X, $\mathbf{0} + x = x = x + \mathbf{0}$.

For example, any linear subspace of $\mathbb{R}^n$ (including the zero subspace, and $\mathbb{R}^n$ itself) is itself a linear space.

(1) Which of the following are linear spaces?

(a) The set of all 2×2 matrices.

(b) The set of all polynomials of degree 3 or less. (i.e., functions of the form $c_3x^3 + c_2x^2 + c_1x + c_0$)

(c) The set of all polynomials of degree 3 or less such that $P(5) = 0$.

(d) The set of all 2×2 matrices with lower left entry equal to 0.

(e) The set of all polynomials of degree 3 or less such that $P(5) = 3$.

(f) The set of all real-valued functions on the interval $[1, 3]$.

(g) The set of all *continuous* real-valued functions on the interval $[1, 3]$.

(h) The set of all *positive* real-valued functions on the interval $[1, 3]$.

(2) For each of the following linear spaces, give the dimension and find a basis (or say in a few words why it is impossible).

(a) The space of all polynomials of degree 3 or less.

(b) The space of all polynomials of degree 3 or less such that $P(5) = 0$.

(c) The space of continuous functions on $[1, 3]$.

(d) The space of 2×3 matrices A such that $A \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

(e) The space of 2×3 matrices A such that $A \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(3) Find a matrix A such that the image of the matrix $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$ coincides with the kernel of the matrix A.