## MATH 21B, FEBRUARY 16: CHANGE OF BASIS

Definition: Let $\mathcal{B}=\left(\vec{v}_{1}, \ldots, \vec{v}_{n}\right)$ be a basis for $\mathbb{R}^{n}$. Then for any $\vec{x} \in \mathbb{R}^{n}$, if $\vec{x}=c_{1} \vec{v}_{1}+\ldots+$ $c_{n} \vec{v}_{n}$, we say the $c_{i}$ 's are the $\mathcal{B}$-coordinates of $\vec{x}$, and define

$$
[\vec{x}]_{\mathcal{B}}=\left[\begin{array}{c}
c_{1} \\
\vdots \\
c_{n}
\end{array}\right]
$$

The $\mathcal{B}$-coordinates of $\vec{x}$ can be obtained by solving the system $S\left([\vec{x}]_{\mathcal{B}}\right)=\vec{x}$, where $S$ is the matrix whose columns are $\vec{v}_{1}, \ldots, \vec{v}_{n}$. The answer has the formula

$$
[\vec{x}]_{\mathcal{B}}=S^{-1}(\vec{x})
$$

(1) Find the $\mathcal{B}$-coordinates of $\vec{x}$, or explain why it cannot be done.
(a) $S=\left[\begin{array}{ccc}1 & -1 & -1 \\ 1 & 1 & 5 \\ -2 & 0 & -3\end{array}\right], \vec{x}=\left[\begin{array}{c}0 \\ 10 \\ -9\end{array}\right]$

Solution: We solve the system

$$
\left[\begin{array}{ccc:c}
1 & -1 & -1 & 0 \\
1 & 1 & 5 & 10 \\
-2 & 0 & -3 & -9
\end{array}\right] \xrightarrow{\text { rref }}\left[\begin{array}{lll:l}
1 & 0 & 0 & 3 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 1
\end{array}\right]
$$

Therefore, $\vec{x}$ has $\mathcal{B}$-coordinates $\left[\begin{array}{l}3 \\ 2 \\ 1\end{array}\right]$. We can verify that

$$
3\left[\begin{array}{c}
1 \\
1 \\
-2
\end{array}\right]+2\left[\begin{array}{c}
-1 \\
1 \\
0
\end{array}\right]+\left[\begin{array}{c}
-1 \\
5 \\
-3
\end{array}\right]=\left[\begin{array}{c}
0 \\
10 \\
-9
\end{array}\right]
$$

(b) $S=\left[\begin{array}{ll}3 & 2 \\ 7 & 5\end{array}\right], \vec{x}=\left[\begin{array}{l}1 \\ 0\end{array}\right]$

Solution: We solve the system

$$
\left[\begin{array}{cc:c}
3 & 2 & 1 \\
7 & 5 & 0
\end{array}\right] \rightarrow\left[\begin{array}{cc:c}
1 & 0 & 5 \\
0 & 1 & -7
\end{array}\right]
$$

Alternatively, we have previously computed that $S^{-1}=\left[\begin{array}{cc}5 & -2 \\ -7 & 3\end{array}\right]$, and so we see that the $\mathcal{B}$-coordinates of $\vec{e}_{1}$ are, by definition, the first column of this matrix.
(c) $S=\left[\begin{array}{lll}1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9\end{array}\right], \vec{x}=\left[\begin{array}{l}1 \\ 1 \\ 2\end{array}\right]$

Solution: The columns of $S$ do not form a basis, as row reduction of $S$ gives a row of all zeroes (and therefore $S$ has nontrivial kernel)! Therefore, it does not make sense to put $\vec{x}$ into $\mathcal{B}$ coordinates. In this particular case, $\vec{x}$ does not lie in the image of $S$, so there is no way to write $\vec{x}$ as a linear combination of the columns of $S$. But even if $\vec{x}$ lay in the image of $S$, there would be infinitely many ways to write $\vec{x}$ as a linear combination of the columns of $S$.
(2) Let $V$ be the plane $x+y-z=0$ in $\mathbb{R}^{3}$. Find a basis for $\mathbb{R}^{3}$ in which every vector of $V$ has the form $\left[\begin{array}{l}a \\ b \\ 0\end{array}\right]$.

Solution: We want to find a basis $\vec{v}_{1}, \vec{v}_{2}, \vec{v}_{3}$ of $\mathbb{R}^{3}$ such that $V=\left\langle\vec{v}_{1}, \vec{v}_{2}\right\rangle . V$ is the set of points perpendicular to $\left[\begin{array}{c}1 \\ 1 \\ -1\end{array}\right]$, i.e.

$$
V=\operatorname{ker}\left[\begin{array}{lll}
1 & 1 & -1
\end{array}\right]
$$

This $1 \times 3$ matrix is already in rref, and has two free variables - namely, $y$ and $z$. Thus,

$$
\operatorname{ker}\left[\begin{array}{lll}
1 & 1 & -1
\end{array}\right]=\left\langle\left[\begin{array}{c}
-1 \\
1 \\
0
\end{array}\right],\left[\begin{array}{l}
1 \\
0 \\
1
\end{array}\right]\right\rangle
$$

which yields our first two vectors $\vec{v}_{1}$ and $\vec{v}_{2}$. To get $\vec{v}_{3}$, we might as well pick any vector outside $V: \vec{v}_{3}=\left[\begin{array}{c}1 \\ 1 \\ -1\end{array}\right]$ is a perfectly good example we found earlier. Thus (this is not the only solution)

$$
S=\left[\begin{array}{ccc}
-1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & -1
\end{array}\right]
$$

Definition: Let $A$ be any $n \times n$ matrix. Then the $\mathcal{B}$-matrix of $A$ is the matrix which tells us what $A$ does to vectors in $\mathcal{B}$-coordinates. It is given by the formula

$$
B=S^{-1} A S
$$

One says that $B$ is similar to $A$.
(3) Let $V$ be the plane $x+y-z=0$ in $\mathbb{R}^{3}$. We are going to write down $A$, the $3 \times 3$ matrix for projection onto $V$.
(a) What is a sensible basis to use as coordinates? (Write down a basis for $\mathbb{R}^{3}$ where we can easily write the projection of each basis element onto $V$.) Call the matrix associated to this basis $S$.
Solution: The basis we found in the last problem is a good choice, because the two vectors $\vec{v}_{1}, \vec{v}_{2}$ spanning $V$ are projected to themselves, and the third vector $\vec{v}_{3}$ is projected to 0 , because it is orthogonal to $V$.
(b) Write down the matrix $B=S^{-1} A S$ for projection onto $V$ in this basis. (Hint: what does projection do to the basis vectors $\vec{v}_{1}, \vec{v}_{2}, \vec{v}_{3}$ ? Your matrix $B$ should be really simple!)
Solution: Projection sends $\vec{v}_{1} \mapsto \vec{v}_{1}, \vec{v}_{2} \mapsto \vec{v}_{2}$, and $\vec{v}_{3} \mapsto 0$. Thus, $B=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]_{\mathcal{B}}$. (I have used a subscript $\mathcal{B}$ to remind myself that we are in $\mathcal{B}$-coordinates)
(c) Now use this to write down the matrix $A$ for projection onto $V$ in the standard basis $\vec{e}_{1}, \vec{e}_{2}, \vec{e}_{3}$.
Solution: We have that $A=S B S^{-1}$, i.e.

$$
A=\left[\begin{array}{ccc}
-1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & -1
\end{array}\right]\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right]\left[\begin{array}{ccc}
-1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & -1
\end{array}\right]^{-1}
$$

So we need to invert the matrix $S$. We do this by standard row-reduction.

$$
\left[\begin{array}{ccc:ccc}
-1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 & 0 & 1
\end{array}\right] \rightarrow\left[\begin{array}{ccc:ccc}
1 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\
0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\
0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3}
\end{array}\right]
$$

and therefore, $S^{-1}=\frac{1}{3}\left[\begin{array}{ccc}-1 & 2 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & -1\end{array}\right]$. We now perform the matrix multiplication

$$
\begin{aligned}
& A=\frac{1}{3}\left[\begin{array}{ccc}
-1 & 1 & 1 \\
1 & 0 & 1 \\
0 & 1 & -1
\end{array}\right]\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right]\left[\begin{array}{ccc}
-1 & 2 & 1 \\
1 & 1 & 2 \\
1 & 1 & -1
\end{array}\right] \\
&=\frac{1}{3}\left[\begin{array}{ccc}
-1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right]\left[\begin{array}{ccc}
-1 & 2 & 1 \\
1 & 1 & 2 \\
1 & 1 & -1
\end{array}\right]=\frac{1}{3}\left[\begin{array}{ccc}
2 & -1 & 1 \\
-1 & 2 & 1 \\
1 & 1 & 2
\end{array}\right]
\end{aligned}
$$

and therefore, $A=\frac{1}{3}\left[\begin{array}{ccc}2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2\end{array}\right]$. As verification, you can check that $A \vec{v}_{1}=$
$\vec{v}_{1}, A \vec{v}_{2}=\vec{v}_{2}, A \vec{v}_{3}=0$.
(4) NOTE: The solution to this problem as originally stated actually ended up being much more involved than I had intended. I explain some of the reasons why in the footnote at the bottom of the page. Nonetheless, for the sake of completeness, I've included the solution here - but think of it as looking slightly ahead in the course, rather than something you need to be able to solve right now!

In this problem, we will write down the matrix $A$ for counterclockwise rotation by $\theta$ around the line $L$ in $\mathbb{R}^{3}$ spanned by the vector $\left[\begin{array}{c}2 \\ 1 \\ -1\end{array}\right]$. (i.e., if you point your thumb of your right hand in the direction of the vector, then the direction that your fingers curl is the direction the rotation will go.) $\square^{1}$
(a) Find a sensible basis $\mathcal{B}$ for this problem. (Hint: start with a basis for the plane perpendicular to this line.)
Solution: Any vector on the line itself is preserved by the rotation, so our first basis vector will be $\vec{v}_{1}=\left[\begin{array}{c}2 \\ 1 \\ -1\end{array}\right]$. The plane perpendicular to $L$ gets mapped to itself by the rotation, so for $\vec{v}_{2}$ and $\vec{v}_{3}$, we will find a basis for this plane. That is, we compute a basis for

$$
\operatorname{ker}\left[\begin{array}{lll}
2 & 1 & -1
\end{array}\right]=\left\langle\left[\begin{array}{c}
-1 / 2 \\
1 \\
0
\end{array}\right],\left[\begin{array}{c}
1 / 2 \\
0 \\
1
\end{array}\right]\right\rangle=\left\langle\left[\begin{array}{c}
-1 \\
2 \\
0
\end{array}\right],\left[\begin{array}{l}
1 \\
0 \\
2
\end{array}\right]\right\rangle
$$

We want rotation in this plane to be easily expressible in terms of the two basis vectors $\vec{v}_{2}$ and $\vec{v}_{3}$, so we should pick them so that they are orthogonal and have the same length.
First, let's make them orthogonal by replacing the second one by $\left[\begin{array}{c}1 / 2 \\ 0 \\ 1\end{array}\right]+\frac{1}{5}\left[\begin{array}{c}-1 / 2 \\ 2 \\ 0\end{array}\right]=$

[^0]$\left[\begin{array}{c}2 / 5 \\ 1 / 5 \\ 1\end{array}\right]$ (the coefficient $\frac{1}{5}$ chosen so that the dot product of this new vector with $\left[\begin{array}{c}-1 / 2 \\ 1 \\ 0\end{array}\right]$ is zero). Now we scale them both so that they become unit vectors.

$$
\left[\begin{array}{c}
2 / 5 \\
1 / 5 \\
1
\end{array}\right] \mapsto\left[\begin{array}{c}
2 / \sqrt{30} \\
1 / \sqrt{30} \\
5 / \sqrt{30}
\end{array}\right]=\vec{v}_{2} \quad\left[\begin{array}{c}
1 / 2 \\
0 \\
1
\end{array}\right] \mapsto\left[\begin{array}{c}
1 / \sqrt{5} \\
0 \\
2 / \sqrt{5}
\end{array}\right]=\vec{v}_{3} 3_{2}^{2}
$$

(b) If $\vec{x}$ has $\mathcal{B}$-coordinates $\left[\begin{array}{l}c_{1} \\ c_{2} \\ c_{3}\end{array}\right]_{\mathcal{B}}$, then what are the $\mathcal{B}$-coordinates after we rotate it by $\theta$ around $L$ ? Write down the matrix which performs this transformation in $\mathcal{B}$-coordinates. (Hint: what are the $\mathcal{B}$-coordinates of $L$, the line we're rotating around?)
Solution: The line $L$ is the set of points with $\mathcal{B}$-coordinates of the form $\left[\begin{array}{c}c_{1} \\ 0 \\ 0\end{array}\right]_{\mathcal{B}}$, and therefore rotation sends $\left[\begin{array}{c}c_{1} \\ 0 \\ 0\end{array}\right]_{\mathcal{B}} \mapsto\left[\begin{array}{c}c_{1} \\ 0 \\ 0\end{array}\right]_{\mathcal{B}}$. The plane perpendicular to $L$ is the set of points with $\mathcal{B}$-coordinates of the form $\left[\begin{array}{c}0 \\ c_{2} \\ c_{3}\end{array}\right]_{\mathcal{B}}$, and rotation sends this point to $\left[\begin{array}{c}0 \\ c_{2} \cos \theta-c_{3} \sin \theta \\ c_{2} \sin \theta+c_{3} \cos \theta\end{array}\right]_{\mathcal{B}}$. Therefore, rotation sends $\left[\begin{array}{c}c_{1} \\ c_{2} \\ c_{3}\end{array}\right]_{\mathcal{B}}$ to $\left[\begin{array}{c}c_{1} \\ c_{2} \cos \theta-c_{3} \sin \theta \\ c_{2} \sin \theta+c_{3} \cos \theta\end{array}\right]_{\mathcal{B}}$, and is therefore described by the matrix

$$
B=[A]_{\mathcal{B}}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{array}\right]_{\mathcal{B}}
$$

[^1](c) Use this to find the matrix $A$ in the standard coordinates which rotates around the line $L$.
Solution: We use the formula
\[

A=S^{-1} B S=\left[$$
\begin{array}{ccc}
2 & 2 / \sqrt{30} & 1 / \sqrt{5} \\
1 & 1 / \sqrt{30} & 0 \\
-1 & 5 / \sqrt{30} & 2 / \sqrt{5}
\end{array}
$$\right]^{-1}\left[$$
\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{array}
$$\right]\left[$$
\begin{array}{ccc}
2 & 2 / \sqrt{30} & 1 / \sqrt{5} \\
1 & 1 / \sqrt{30} & 0 \\
-1 & 5 / \sqrt{30} & 2 / \sqrt{5}
\end{array}
$$\right]
\]

You can compute by standard row reduction that

$$
S^{-1}=\left[\begin{array}{ccc}
\frac{1}{3} & -\frac{5}{3} & -\frac{1}{3} \\
-\frac{2 \sqrt{5}}{\sqrt{6}} & \frac{5 \sqrt{5}}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} \\
\sqrt{5} & -2 \sqrt{5} & 0
\end{array}\right]
$$

so now it's a big matrix multiplication. Blegh, I don't feel like writing it out... in principle, for any given $\theta$ we can compute it.


[^0]:    ${ }^{1}$ WARNING: This problem ended up being FAR more involved than I had intended. The reason is as follows: rotation by $\theta$ in $\mathbb{R}^{2}$ is described by the matrix $\left[\begin{array}{cc}\cos \theta & -\sin \theta \\ \sin \theta & \cos \theta\end{array}\right]$, and this same matrix describes the rotation in any basis where our basis vectors are orthogonal and have the same length. However, if our basis vectors have different lengths, then the matrix for rotation by $\theta$ is more complicated. And if they are not orthogonal, it's horrendous! Therefore, in picking our basis in this problem, we have to ensure the vectors spanning the plane perpendicular to $L$ are orthogonal and have the same length. I detail how to do this in part (a) - then the rest of the problem is as I intended. The problem also requires that we compute the sign of the determinant to ensure that we are rotating in the correct direction around the line - we will not get to determinants for quite a while, so I am going to sweep that part under the rug, in the interest of minimizing confusion. But I will revisit this problem when we get there!

[^1]:    ${ }^{2}$ I chose the vectors to be ordered in this way so that the determinant of the $3 \times 3$ matrix $\left[\begin{array}{lll}\vec{v}_{1} & \vec{v}_{2} & \vec{v}_{3}\end{array}\right]$ is positive. Don't worry right now about why this is true - when we get to determinants, all will be clear!

