MATH 21B, FEBRUARY 16: CHANGE OF BASIS

Definition: Let $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ be a basis for $\mathbb{R}^n$. Then for any $\vec{x} \in \mathbb{R}^n$, if $\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$, we say the c_i's are the $\mathcal{B}$-coordinates of $\vec{x}$, and define

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

The $\mathcal{B}$-coordinates of $\vec{x}$ can be obtained by solving the system $S([\vec{x}]_{\mathcal{B}}) = \vec{x}$, where S is the matrix whose columns are $\vec{v}_1, \ldots, \vec{v}_n$. The answer has the formula

$$[\vec{x}]_{\mathcal{B}} = S^{-1}(\vec{x})$$

(1) Find the $\mathcal{B}$-coordinates of $\vec{x}$, or explain why it cannot be done.

(a) $S = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 5 \\ -2 & 0 & -3 \end{bmatrix}, \vec{x} = \begin{bmatrix} 0 \\ 10 \\ -9 \end{bmatrix}$

(b) $S = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}, \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(c) $S = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

(2) Let V be the plane $x+y-z=0$ in $\mathbb{R}^3$. Find a basis for $\mathbb{R}^3$ in which every vector of V has the form $\begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$.

Definition: Let A be any $n \times n$ matrix. Then the $\mathcal{B}$-matrix of A is the matrix which tells us what A does to vectors in $\mathcal{B}$-coordinates. It is given by the formula

$$B = S^{-1}AS$$

One says that B is *similar* to A.

(3) Let V be the plane $x+y-z=0$ in $\mathbb{R}^3$. We are going to write down A, the 3×3 matrix for projection onto V.

(a) What is a sensible basis to use as coordinates? (Write down a basis for $\mathbb{R}^3$ where we can easily write the projection of each basis element onto V.) Call the matrix associated to this basis S.

(b) Write down the matrix $B = S^{-1}AS$ for projection onto V in this basis.

(c) Now use this to write down the matrix A for projection onto V in the standard basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$. (Hint: if $B = S^{-1}AS$, then how do you get A back from B?)

(4) In this problem, we will write down the matrix A for counterclockwise rotation by θ around the line L in $\mathbb{R}^3$ spanned by the vector $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. (i.e., if you point your thumb of your right hand in the direction of the vector, then the direction that your fingers curl is the direction the rotation will go.)

(a) Find a sensible basis $\mathcal{B}$ for this problem. (Hint: start with a basis for the plane perpendicular to this line.)

(b) If $\vec{x}$ has $\mathcal{B}$-coordinates $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, then what are the $\mathcal{B}$-coordinates after we rotate it by θ around L? Write down the matrix which performs this transformation in $\mathcal{B}$-coordinates. (Hint: what are the $\mathcal{B}$-coordinates of L, the line we're rotating around?)

(c) Use this to find the matrix A in the standard coordinates which rotates around the line L.