MATH 21B, FEBRUARY 14: BASIS, LINEAR INDEPENDENCE, AND DIMENSION

Definition: A sequence of vectors $\vec{v}_1, \dots, \vec{v}_m$ is called *linearly independent* if there are *no nontrivial linear relations*
$$a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = 0$$
unless $a_1 = a_2 = \cdots = a_m = 0$.

Definition: A subset V of $\mathbb{R}^n$ is said to be a *linear subspace* if:
- $0 \in V$.
- $\vec{v} + \vec{w} \in V$ whenever $\vec{v}, \vec{w} \in V$. (Closed under addition)
- $\lambda\vec{v} \in V$ whenever $\vec{v} \in V$ and λ is a real number. (Closed under scalar multiplication)

Definition: Let V be a linear subspace of $\mathbb{R}^n$. A set of vectors $\vec{v}_1, \dots, \vec{v}_m$ is a *basis* of V if they span V and are linearly independent. In this case, every vector of V can be expressed *uniquely* as a linear combination of $\vec{v}_1, \dots, \vec{v}_m$.

(1) Which of the following sets of vectors are linear subspaces?

(a) The union of the x- and y-axes in $\mathbb{R}^2$.
No: not closed under addition.

(b) The kernel of a matrix.
Yes.

(c) The image of a matrix.
Yes.

(d) The plane $x + 2y - z = 2$ in $\mathbb{R}^3$.
No: doesn't contain $\vec{0}$.

(e) The span of a collection of vectors in $\mathbb{R}^n$.
Yes.

(f) The set $x > 0$ in $\mathbb{R}^2$ (called the *upper half plane*).
No: doesn't contain $\vec{0}$ and not closed under scaling.

(2) Which of the following sequences of vectors are linearly independent?

(a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \end{bmatrix}$
No: $3\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
No: $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}2\\3\\-1\end{bmatrix}$

No: $2\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} - 1\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\3\\-1\end{bmatrix}$

(d) $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}$

Yes. In particular, they span $\mathbb{R}^3$.

(3) Consider the matrix $A = \begin{bmatrix}1 & 0 & 2\\3 & 1 & 7\\1 & 2 & 4\\-1 & 7 & 5\end{bmatrix}$

(a) Compute $\operatorname{rref}(A)$, and use this to compute $\ker(A)$.
Solution: By standard row reduction,

$$\operatorname{rref}(A) = \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

The kernel of this matrix is the same as the kernel of A, because row operations just correspond to adding and scaling equations. Finding the kernel of this matrix corresponds to solving the equations $x_1 + 2x_3 = 0$ and $x_2 + x_3 = 0$. x_3 is the *free variable*, and x_1, x_2 are the leading variables whose values are $-2x_3$ and $-x_3$, respectively. Therefore,

$$\ker(\operatorname{rref}(A)) = \left\{ \begin{bmatrix}-2t\\-t\\t\end{bmatrix} : t \in \mathbb{R} \right\}$$

(b) Write a basis for $\ker(A)$.
Solution: There is only one free variable, so $\ker(A)$ has only one basis vector, namely $\begin{bmatrix}-2\\-1\\1\end{bmatrix}$.

(c) Write $\operatorname{im}(A)$ as the span of a collection of vectors.
Solution: The image of A is just the span of the columns, because the columns are the images of $\vec{e}_1, \vec{e}_2, \vec{e}_2$. Thus,

$$\operatorname{im}(A) = \left\langle \begin{bmatrix}1\\3\\1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\2\\7\end{bmatrix}, \begin{bmatrix}2\\7\\4\\5\end{bmatrix} \right\rangle$$

(d) Are these vectors linearly independent? If not, can you write down a linear relation? (Hint: use the kernel!)

Solution: No, they are not. We know that since $\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$ is in the kernel, that $A\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. This tells us that

$$-2\begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 2 \\ 7 \end{bmatrix} + \begin{bmatrix} 2 \\ 7 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(e) Write down a basis for $\mathrm{im}(A)$.

Solution: Any two of the three vectors above form a basis for $\mathrm{im}(A)$. For example, $\begin{bmatrix} 1 \\ 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 7 \end{bmatrix}$ is a basis.

General steps to find a basis for $\ker(A)$**:**

(a) Row reduce A to get $\mathrm{rref}(A)$. Clearly, $\mathrm{rref}(A)$ has the same kernel as A.

(b) Decide which are the free variables and which are the leading variables. Let's call the free variables $t_1, \ldots, t_s$ (where s is the nullity).

(c) For each of the free variables, set it equal to 1 and all of the other free variables equal to 0. This will determine the values of the leading variables, and will thus give you a vector.

(d) Do this for all s free variables, and you get s vectors. These vectors form a basis for the kernel.

General steps to find a basis for $\mathrm{im}(A)$**:**

(a) Row reduce A to get $\mathrm{rref}(A)$. ($\mathrm{rref}(A)$ does *not* have the same image as A!)

(b) Decide which are the free variables and which are the leading variables. Each leading variable corresponds to a column.

(c) For each leading variable, take the corresponding column of A. These vectors form a basis for the image.

Why does this second algorithm work? This is best demonstrated through an example - I will use 4(b) to show this.

(4) For each of the following matrices, calculate a basis for its kernel and image. (Since you know how to row-reduce, I have included $\mathrm{rref}(A)$ for each matrix.)

(a) $A = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 5 \end{bmatrix}$, $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

Solution: The first three columns are leading, and the last column is free. I've used the colors red and blue to show this. For the kernel: let's set the free variable x_4 to be equal to 1. The three equations corresponding to the rows tell us that $x_1 = -1, x_2 = -2, x_3 = -3$. Therefore, the kernel has one basis vector, $\begin{bmatrix} -1 \\ -2 \\ -3 \\ 1 \end{bmatrix}$. For the image: we take the first three rows of A, so the image has basis $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ -1 & -1 & -2 & 3 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 2 & -1 \end{bmatrix}$, $\mathrm{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution: For the kernel: set the free variable x_2 to be 1 to get $x_1 = -1, x_3 = 0, x_4 = 0$. Therefore, the kernel has one basis vector, $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. The image has basis given by the column vectors $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ -1 \end{bmatrix}$. Now, I owe you an explanation of why this method for the image works. Focus on the first, third, and fourth columns of the matrices above. The row reduction does the following.

$$\begin{bmatrix} 1 & 3 & -1 \\ -1 & -2 & 3 \\ 1 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly, the matrix on the right has kernel equal to $\{\vec{0}\}$ (i.e., *trivial* kernel), because it is just the identity matrix with a row of zeroes at the bottom (in general, if you just select the leading columns of $\mathrm{rref}(A)$ and make a matrix out of those, you'll get the identity matrix plus some rows of zeroes at the bottom). Therefore, the matrix on the left has trivial kernel as well (because they have the same kernel). A matrix with trivial kernel has linearly independent columns (this is essentially just by definition). Therefore, the matrix on the left, formed by the leading columns of A, has linearly independent columns.

(c) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 5 \\ -2 & 0 & -3 \end{bmatrix}$, $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: There are no free variables, so $\ker(A) = \{\vec{0}\}$ has no basis. The image has basis equal to all three columns - in particular, this means the image is $\mathbb{R}^3$, and so we could also say it has basis $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

(5) True or false: if A is a 5×4 matrix with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$, and if $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \ker(A)$, then $\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = \vec{0}$.

Solution: True! Check for yourself that the following matrix times vector equation holds

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4$$

where the matrix on the left is the 5×4 matrix A whose columns are $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$. (It is essentially because $A\vec{e}_i = \vec{v}_i$ for $i = 1, 2, 3, 4$.)

Definition: If $V \subset \mathbb{R}^n$ is a linear subspace, then it has a basis $\vec{v}_1, \ldots, \vec{v}_m$, and the size m of the basis is *independent of which basis we pick.* We call this number m the *dimension* of V. In particular, if V is spanned by n vectors, then its dimension is at most n.

For any matrix A, we call the dimension of $\text{im}(A)$ the *rank*, and we call the dimension of $\ker(A)$ the *nullity*. The rank equals the number of leading variables, and the nullity equals the number of free variables. Thus, we have the **rank-nullity theorem**, which states that rank + nullity = number of columns.

(6) True or false?

(a) If A is a 4×3 matrix and $A\vec{x} = \vec{0}$ has no nonzero solutions, then the columns of A are linearly independent.

Solution: **True.** In general, for *any* matrix A, $A\vec{x} = \vec{0}$ has no solutions if and only if the columns of A are linearly independent.

(b) If A is a 4×3 matrix and $A\vec{x} = \vec{0}$ has no nonzero solutions, then $A\vec{x} = \vec{e}_1$ has a solution.

Solution: **False.** For example, consider

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix has linearly independent columns, so its columns span a 3-dimensional linear subspace of $\mathbb{R}^4$, but in this case, that subspace does not contain $\vec{e}_1$.

(c) There is a 3×6 matrix whose kernel is two-dimensional.
Solution: False. The columns of a 3×6 matrix live in $\mathbb{R}^3$, and thus the rank is at most 3. Since rank + nullity = 6, the nullity is at least 3. (In other words: there are six variables, and at most three of them are leading, therefore at least three are free variables and so the kernel has dimension at least 3.)

(d) If A is a 3×5 matrix whose kernel is two-dimensional, then $A\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ has a unique solution.
Solution: False. Rank + nullity = 5, so the rank is 3. Therefore, the image is all of $\mathbb{R}^3$, and so $A\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ does have a solution. However, this solution is *not unique*, because if $\vec{x}$ is one solution, and $\vec{v}$ is any element of $\ker(A)$, then $A(\vec{x} + \vec{v}) = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

(e) There exists a 5×4 matrix whose image is $\mathbb{R}^5$.
Solution: False. The image of a 5×4 vector has dimension 4 or smaller - but $\mathbb{R}^5$ has dimension 5.