MATH 21B, FEBRUARY 14: BASIS, LINEAR INDEPENDENCE, AND DIMENSION

Definition: A sequence of vectors $\vec{v}_1, \dots, \vec{v}_m$ is called *linearly independent* if there are *no nontrivial linear relations*

$$a_1\vec{v}_1 + \ldots + a_m\vec{v}_m = 0$$

unless $a_1 = a_2 = \cdots = a_m = 0$.

Definition: A subset V of $\mathbb{R}^n$ is said to be a *linear subspace* if:
- $0 \in V$.
- $\vec{v} + \vec{w} \in V$ whenever $\vec{v}, \vec{w} \in V$.
- $\lambda\vec{v} \in V$ whenever $\vec{v} \in V$ and λ is a real number.

Definition: Let V be a linear subspace of $\mathbb{R}^n$. A set of vectors $\vec{v}_1, \dots, \vec{v}_m$ is a *basis* of V if they span V and are linearly independent. In this case, every vector of V can be expressed *uniquely* as a linear combination of $\vec{v}_1, \dots, \vec{v}_m$.

(1) Which of the following sets of vectors are linear subspaces?

(a) The union of the x- and y-axes in $\mathbb{R}^2$.

(b) The kernel of a matrix.

(c) The image of a matrix.

(d) The plane $x + 2y - z = 2$ in $\mathbb{R}^3$.

(e) The span of a collection of vectors in $\mathbb{R}^n$.

(f) The set $x > 0$ in $\mathbb{R}^2$ (called the *upper half plane*).

(2) Which of the following sequences of vectors are linearly independent?

(a) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(3) Consider the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 7 \\ 1 & 2 & 4 \\ -1 & 7 & 5 \end{bmatrix}$

(a) Compute $\mathrm{rref}(A)$, and use this to compute $\ker(A)$.

(b) Write a basis for $\ker(A)$.

(c) Write $\mathrm{im}(A)$ as the span of a collection of vectors.

(d) Are these vectors linearly independent? If not, can you write down a linear relation? (Hint: use the kernel!)

(e) Write down a basis for $\mathrm{im}(A)$.

(4) For each of the following matrices, calculate a basis for its kernel and image. (Since you know how to row-reduce, I have included rref(A) for each matrix.)

(a) $A = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & 5 \end{bmatrix}$, $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 3 & -1 \\ -1 & -1 & -2 & 3 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 2 & -1 \end{bmatrix}$, $\operatorname{rref}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 5 \\ -2 & 0 & -3 \end{bmatrix}$, $\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(5) True or false: if A is a 5×4 matrix with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$, and if $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \ker(A)$, then $\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = \vec{0}$.

Definition: If $V \subset \mathbb{R}^n$ is a linear subspace, then it has a basis $\vec{v}_1, \ldots, \vec{v}_m$, and the size m of the basis is *independent of which basis we pick.* We call this number m the *dimension* of V. In particular, if V is spanned by n vectors, then its dimension is at most n.

For any matrix A, we call the dimension of $\text{im}(A)$ the *rank*, and we call the dimension of $\ker(A)$ the *nullity*. The rank equals the number of leading variables, and the nullity equals the number of free variables. Thus, we have the **rank-nullity theorem**, which states that rank + nullity = number of columns.

(6) True or false?

(a) If A is a 4×3 matrix and $A\vec{x} = \vec{0}$ has no nonzero solutions, then the columns of A are linearly independent.

(b) If A is a 4×3 matrix and $A\vec{x} = \vec{0}$ has no nonzero solutions, then $A\vec{x} = \vec{e}_1$ has a solution.

(c) There is a 3×6 matrix whose kernel is two-dimensional.

(d) If A is a 3×5 matrix whose kernel is two-dimensional, then $A\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ has a unique solution.

(e) There exists a 5×4 matrix whose image is $\mathbb{R}^5$.