MATH 21B, FEBRUARY 9: MATRIX INVERSION, IMAGE, KERNEL, AND RANK

Given an $n \times p$ matrix A, and a $p \times m$ matrix B, we define the *matrix product* $A \cdot B$ as follows. If B has column vectors

$$B = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix}$$

then $A \cdot B$ (or just AB) is the $n \times m$ matrix

$$AB = \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_m \end{bmatrix}$$

A represents a linear transformation $\mathbb{R}^p \to \mathbb{R}^n$, and B represents a linear transformation $\mathbb{R}^m \to \mathbb{R}^p$; $A \cdot B$ represents the *composition*, which is a linear transformation $\mathbb{R}^m \to \mathbb{R}^n$. Recall that the $n \times n$ matrix with 1's along the diagonal and 0's everywhere else is called the *identity matrix*, and is denoted I_n.

Suppose that A is an $n \times n$ matrix such that for *every* vector $\vec{y} \in \mathbb{R}^n$, there is *exactly one* vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{y}$. Then A has an *inverse* matrix, denoted A^{-1}, such that

$$A\vec{x} = \vec{y} \iff \vec{x} = A^{-1}\vec{y}$$

(1) In this problem, we will invert the matrix $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$.

(a) Show that $\mathrm{rref}(A) = I_2$.
Solution: Standard row reduction. See part (b) for the reduction.

(b) Row-reduce the augmented matrix $[A|I_2] = \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 7 & 5 & 0 & 1 \end{array}\right]$ to get $[I_2|A^{-1}]$.
Solution:

$$\left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 7 & 5 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2/3 & 1/3 & 0 \\ 7 & 5 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2/3 & 1/3 & 0 \\ 0 & 1/3 & -7/3 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & 5 & -2 \\ 0 & 1/3 & -7/3 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & 5 & -2 \\ 0 & 1 & -7 & 3 \end{array}\right]$$

so $A^{-1} = \begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$.

(c) What is $A \cdot A^{-1}$? What about $A^{-1} \cdot A$?
Solution: Simply multiplying matrices, we find $AA^{-1} = I_2$, and also $A^{-1}A = I_2$. These properties are an alternative way to defined the inverse A^{-1} of the matrix A.

(2) Recall that the matrix for a counterclockwise rotation by θ in $\mathbb{R}^2$ is $A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. How do you know this matrix is invertible? What is its inverse?

Solution: The inverse to this matrix should be rotation *clockwise* by θ, i.e., rotation by $-\theta$. Therefore, its matrix is

$$A_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

(3) In worksheet 4 #2(b), you found the matrix A for projection onto the line $y = 3x$ was $A = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$. Does this matrix have an inverse? How do you know?

Solution: No, this matrix has no inverse. In order for a function from one set to another to have an inverse, it must be *one-to-one* (no two inputs give the same output) and *onto* (every element of the range is in the output). In the case of the linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$, it is not one-to-one: every point on the line $y = -\frac{1}{3}x$ is sent to the same output, $(0, 0)$. Therefore, A cannot have an inverse.

(4) If $n \times n$ matrices A and B are invertible, what about AB? If so, give its inverse.

Solution: Yes, $(AB)^{-1} = B^{-1} \cdot A^{-1}$. This can be easily verified by checking what happens when we multiply this matrix by AB.

$$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_2A^{-1} = AA^{-1} = I_2$$

$$(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}I_2B = B^{-1}B = I_2$$

(5) Is the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ invertible?

No, it is not, because rref $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, but in order for a 3×3 matrix to be invertible, its reduced row echelon form must be I_3 (i.e., have 3 leading ones).

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. The *kernel* of T is the set of vectors $\vec{x} \in \mathbb{R}^m$ such that

$$T(\vec{x}) = \vec{0}$$

The *image* of T is the set of vector $\vec{y} \in \mathbb{R}^n$ for which there exists some $\vec{x} \in \mathbb{R}^m$ with

$$T(\vec{x}) = \vec{y}$$

Note that the kernel lives in the *domain*, while the image lives in the *range*. The *span* of a set of vectors $\vec{v}_1, \ldots, \vec{v}_m$ is the set of all vectors which can be written as a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$, and is denoted

$$\langle \vec{v}_1, \ldots, \vec{v}_m \rangle$$

Note that for any matrix A, each of the column vectors of A is in the image (because these are the images of $\vec{e}_1, \vec{e}_2, \ldots$. It follows that the image of A is given by all *linear combinations* of the column vectors - i.e., it is the span of the column vectors.

(6) For each of the following matrices, compute the kernel and image.

(a) $A = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

Solution: The image is the span $\langle \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \end{bmatrix} \rangle$. Since $\begin{bmatrix} 3 \\ 9 \end{bmatrix}$ is a multiple of $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$, the second vector is redundant, and so the image just equals the span $\langle \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rangle$.

We get the kernel by finding the general solution to

$$\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The general solution is $\begin{bmatrix} -3t \\ t \end{bmatrix}$ where $t \in \mathbb{R}$. Another way to write this is as the span $\langle \begin{bmatrix} -3 \\ 1 \end{bmatrix} \rangle$.

(b) $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

Solution: The image is generated by the columns, and is thus equal to $\langle \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \rangle$. Note that since

$$5 \begin{bmatrix} 3 \\ 7 \end{bmatrix} - 7 \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{e}_1$$

$$-2 \begin{bmatrix} 3 \\ 7 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{e}_2$$

it follows that this span contains $\vec{e}_1$ and $\vec{e}_2$ and therefore is all of $\mathbb{R}^2$. For the kernel, just like last time, we solve the equation $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \vec{x} = \vec{0}$, and find that the only solution is $\vec{0}$, so the kernel is $\{\vec{0}\}$.

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution: The image is the span of the column vectors. To determine *which columns are redundant*, we perform row-reduction

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \overset{\text{rref}}{\to} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The *columns with leading ones* correspond to the columns which generate the image of our original matrix A - i.e., the first two columns.

$$\operatorname{im}(A) = \left\langle \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\rangle = \left\{ c_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}$$

To calculate the kernel, we solve the system $A\vec{x} = \vec{0}$. Using row-reduction, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and so $x_1 = x_3$, $x_2 = -2x_3$. Thus, the general solution is

$$\ker(A) = \left\{ \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} : t \in \mathbb{R} \right\} = \left\langle \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\rangle$$

(d) $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{bmatrix}$

Solution: Row-reduce.

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{bmatrix} \overset{\text{rref}}{\to} \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Since only the first column has a leading 1, it follows that the first column of A spans the image (indeed, you can see that the second column is a multiple of the first). So

$$\operatorname{im}(A) = \left\langle \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} \right\rangle$$

The kernel of A equals the kernel of $\text{rref}(A)$ (as we have done in the previous problems), which is the set of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ with $x_1 + (1/2)x_2 = 0$. Therefore

$$\ker(A) = \left\langle \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\rangle$$

(7) If L is a line in $\mathbb{R}^n$ and A is the $n \times n$ matrix for orthogonal projection onto L, then what are the image and kernel of A?

Solution: A sends each vector in $\mathbb{R}^n$ to a point on L, so $\text{im}(A)$ is a subset of L. The points on L are all kept fixed, and so every point of L is in $\text{im}(A)$. Therefore, $\text{im}(A) = L$. The kernel is the set of vectors whose projection onto L is the zero vector: these are the vectors lying on the $(n-1)$-dimensional hyperplane perpendicular to L.

Explicitly in coordinates: if L is spanned by some vector $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, then $\text{im}(A)$ is the span of this vector, and $\ker(A)$ is the hyperplane

$$\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0 \right\}$$

(8) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

(a) Use Gauss-Jordan elimination to write the set of all $\vec{y}$ such that $A\vec{y} = \begin{bmatrix} 6 \\ 15 \\ 24 \end{bmatrix}$.

Solution: Standard row-reduction of the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 15 \\ 7 & 8 & 9 & 24 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

so $x_1 - x_3 = 0 \implies x_1 = x_3$ and $x_2 + 2x_3 = 3 \implies x_2 = -2x_3 + 3$ which gives the general solution

$$\left\{ \begin{bmatrix} t \\ -2t+3 \\ t \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \right\}$$

(b) If $\vec{y}$ and $\vec{z}$ are two such vectors, what can you say about $A(\vec{y} - \vec{z})$?

Solution: If $\vec{y}$ and $\vec{z}$ are two such vectors, then clearly $\vec{y}-\vec{z}$ is a multiple of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, i.e. *the difference lies in the kernel*, so $A(\vec{y}-\vec{z}) = \vec{0}$. (In general, if $\vec{y}$ is any solution to the original equation, we can add any element of the kernel to $\vec{y}$ to get another solution.)