MATH 21B, FEBRUARY 9: MATRIX INVERSION, IMAGE, KERNEL, AND RANK

Given an $n \times p$ matrix A, and a $p \times m$ matrix B, we define the *matrix product* $A \cdot B$ as follows. If B has column vectors

$$B = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix}$$

then $A \cdot B$ (or just AB) is the $n \times m$ matrix

$$AB = \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_m \end{bmatrix}$$

A represents a linear transformation $\mathbb{R}^p \to \mathbb{R}^n$, and B represents a linear transformation $\mathbb{R}^m \to \mathbb{R}^p$; $A \cdot B$ represents the *composition*, which is a linear transformation $\mathbb{R}^m \to \mathbb{R}^n$. Recall that the $n \times n$ matrix with 1's along the diagonal and 0's everywhere else is called the *identity matrix*, and is denoted I_n.

Suppose that A is an $n \times n$ matrix such that for *every* vector $\vec{y} \in \mathbb{R}^n$, there is *exactly one* vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{y}$. Then A has an *inverse* matrix, denoted A^{-1}, such that

$$A\vec{x} = \vec{y} \iff \vec{x} = A^{-1}\vec{y}$$

(1) In this problem, we will invert the matrix $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$.

(a) Show that $\operatorname{rref}(A) = I_2$.

(b) Row-reduce the augmented matrix $[A|I_2] = \left[\begin{array}{cc|cc} 3 & 2 & 1 & 0 \\ 7 & 5 & 0 & 1 \end{array}\right]$ to get $[I_2|A^{-1}]$.

(c) What is $A \cdot A^{-1}$? What about $A^{-1} \cdot A$?

(2) Recall that the matrix for a counterclockwise rotation by θ in $\mathbb{R}^2$ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. How do you know this matrix is invertible? What is its inverse?

(3) In worksheet 4 #2(b), you found the matrix A for projection onto the line $y = 3x$ was $A = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$. Does this matrix have an inverse? How do you know?

(4) If $n \times n$ matrices A and B are invertible, what about AB? If so, give its inverse.

(5) Is the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ invertible?

Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. The *kernel* of T is the set of vectors $\vec{x} \in \mathbb{R}^m$ such that

$$T(\vec{x}) = \vec{0}$$

The *image* of T is the set of vector $\vec{y} \in \mathbb{R}^n$ for which there exists some $\vec{x} \in \mathbb{R}^m$ with

$$T(\vec{x}) = \vec{y}$$

The *span* of a set of vectors $\vec{v}_1, \ldots, \vec{v}_m$ is the set of all vectors which can be written as a linear combination of $\vec{v}_1, \ldots, \vec{v}_m$, and is denoted

$$\langle \vec{v}_1, \ldots, \vec{v}_m \rangle$$

(6) For each of the following matrices, compute the kernel and image.

(a) $\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 1 \\ 4 & 2 \\ -2 & -1 \end{bmatrix}$

(7) If L is a line in $\mathbb{R}^n$ and A is the $n \times n$ matrix for orthogonal projection onto L, then what are the image and kernel of A?

(8) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

(a) Use Gauss-Jordan elimination to write the set of all $\vec{y}$ such that $A\vec{y} = \begin{bmatrix} 6 \\ 15 \\ 24 \end{bmatrix}$.

(b) If $\vec{y}$ and $\vec{z}$ are two such vectors, what can you say about $A(\vec{y} - \vec{z})$?