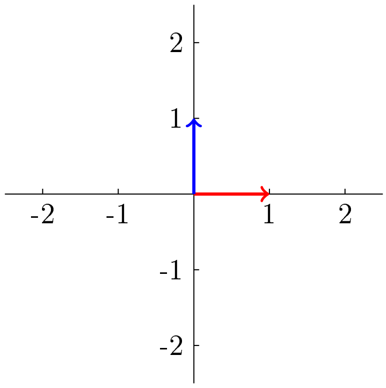
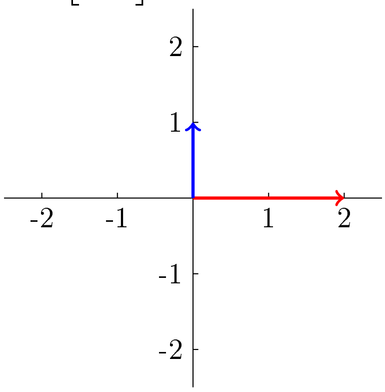
MATH 21B, FEBRUARY 7: ROTATIONS, REFLECTIONS, DILATIONS, PROJECTIONS, AND SHEARS

(1) Consider the vectors $\vec{e}_1$ and $\vec{e}_2$ in $\mathbb{R}^2$.



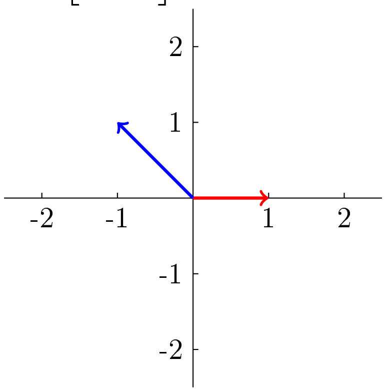
In each part below, you are given a matrix A. Draw what happens to the vectors $\vec{e}_1$ and $\vec{e}_2$ after applying the linear transformation $T(\vec{x}) = A\vec{x}$. Describe the effect of the linear transformation in words.

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$



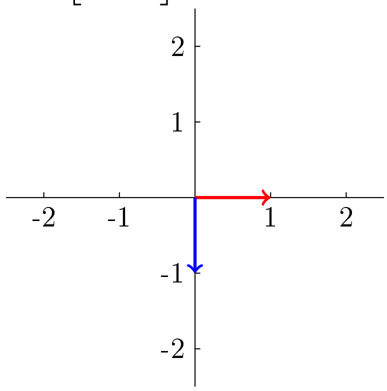
This is a *dilation* by a factor of 2 along the x-axis.

(b) $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$



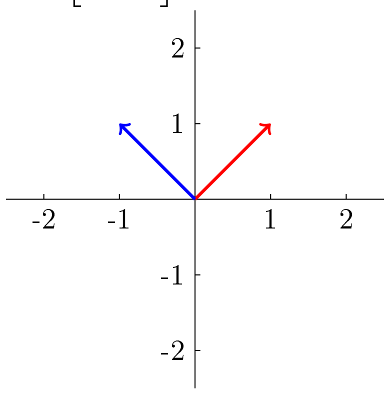
This is a horizontal *shear*.

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



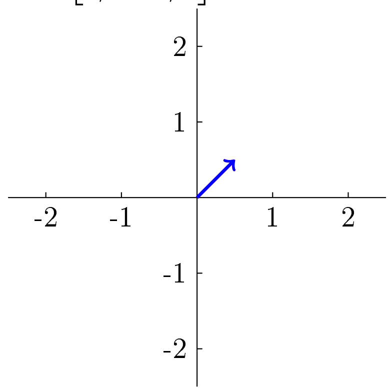
This is a *reflection* over the x-axis. (This is equivalently a dilation by a factor of -1 along the y-axis.)

(d) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$



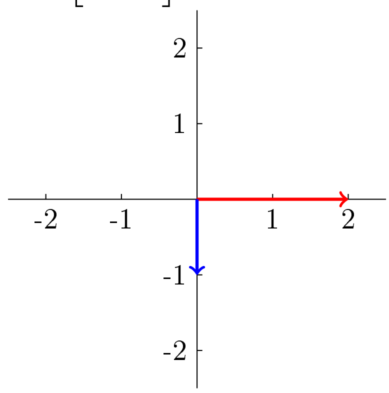
This is a *rotation* counterclockwise by $45°$, followed by a *dilation* by $\sqrt{2}$.

(e) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$



This is a *projection* onto the line $y = x$. (This is equivalently a dilation of factor 0 along the line $y = -x$.)

(f) $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

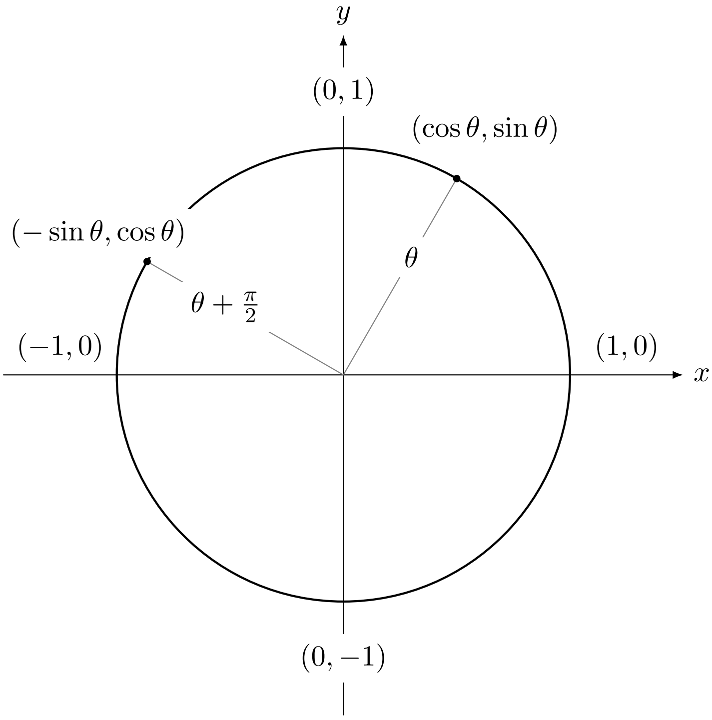


This is a dilation by a factor of 2 along the x-axis composed with a reflection over the x-axis.

(2) For each of the following linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, write down its matrix.

(a) Rotation counterclockwise by an angle θ around the origin. (Where does this transformation send the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?)

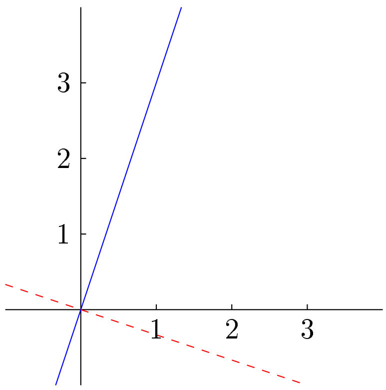
Solution: The first column of this matrix will be the vector that e_1 is sent to. This is, by definition, the point $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$. Similarly, we can see that rotating e_2 counterclockwise by θ gives the point $\begin{bmatrix}\cos(\theta+\pi/2)\\ \sin(\theta+\pi/2)\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$.



Therefore, the desired matrix is

$$A = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

(b) Orthogonal projection onto the line $y = 3x$.
Solution 1:



Instead of calculating the projections of e_1 and e_2, we calculate the projections of some easier vectors. We know that any vector on the line $y = 3x$ (drawn in blue) must get

sent to itself: therefore, $A\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$. We also know that any vector *orthogonal* to the line (drawn in red) gets sent to zero: therefore, $A\begin{bmatrix}3\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$.
We can now use these values to obtain the value of A at e_1 and e_2, because

$$e_1 = \frac{1}{10}\begin{bmatrix}1\\3\end{bmatrix} + \frac{3}{10}\begin{bmatrix}3\\-1\end{bmatrix} \implies Ae_1 = \frac{1}{10}\begin{bmatrix}1\\3\end{bmatrix}$$

$$e_1 = \frac{3}{10}\begin{bmatrix}1\\3\end{bmatrix} - \frac{1}{10}\begin{bmatrix}3\\-1\end{bmatrix} \implies Ae_1 = \frac{3}{10}\begin{bmatrix}1\\3\end{bmatrix}$$

and therefore, $\boxed{A = \begin{bmatrix}1/10 & 3/10\\3/10 & 9/10\end{bmatrix}}$.

Note: In the language of matrix multiplication, which is introduced later in this worksheet, we first obtained the *matrix equation*

$$A\begin{bmatrix}1 & 3\\3 & -1\end{bmatrix} = \begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}$$

We will later see that the system of linear equations we solved was equivalent to finding the *inverse* of the matrix $\begin{bmatrix}1 & 3\\3 & -1\end{bmatrix}$.

Solution 2: We want to calculate the projections of the vectors e_1, e_2 onto this line. We can do so using the *dot product.* Recall that if v and w are two vectors, then the projection of v onto the line defined by w is calculated by the expression

$$\text{proj}_w(v) = \frac{(v \cdot w)w}{|w|^2}$$

In our particular case, the line is defined by the vector $w = \begin{bmatrix}1\\3\end{bmatrix}$, so

$$\text{proj}_w(e_1) = \frac{w}{|w^2|} = \begin{bmatrix}1/10\\3/10\end{bmatrix}$$

$$\text{proj}_w(e_2) = \frac{3w}{|w^2|} = \begin{bmatrix}3/10\\9/10\end{bmatrix}$$

and therefore, the desired matrix is

$$\boxed{A = \begin{bmatrix}1/10 & 3/10\\3/10 & 9/10\end{bmatrix}}$$

(c) Orthogonal projection onto the line $y = 3x$, followed by dilation by a factor of 2 (i.e. double the length of all vectors).

Solution: The images of e_1 and e_2 under this transformation should be double their images under $\begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}$. Therefore, the images are $\begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix}$ and $\begin{bmatrix} 3/5 \\ 9/5 \end{bmatrix}$, and so the required matrix is

$$\boxed{A = \begin{bmatrix} 1/5 & 3/5 \\ 3/5 & 9/5 \end{bmatrix}}$$

Note: In the language of matrix multiplication: dilation by a factor of 2 is given by the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, so the desired composition is given by

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix} = \begin{bmatrix} 1/5 & 3/5 \\ 3/5 & 9/5 \end{bmatrix}$$

Note that in a composition of matrices, the matrix on the right is applied first! (This is similar to composition of functions.)

Given an $n \times p$ matrix A, and a $p \times m$ matrix B, we define the *matrix product* $A \cdot B$ as follows. If B has column vectors

$$B = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix}$$

then $A \cdot B$ (or just AB) is the $n \times m$ matrix

$$AB = \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_m \end{bmatrix}$$

A represents a linear transformation $\mathbb{R}^p \to \mathbb{R}^n$, and B represents a linear transformation $\mathbb{R}^m \to \mathbb{R}^p$; $A \cdot B$ represents the *composition*, which is a linear transformation $\mathbb{R}^m \to \mathbb{R}^n$.

(3) Let $A = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 \\ 5 & 1 \end{bmatrix}$. Find AB and BA (if they make sense).

Solution: A and B are both 2×2 matrices, so we can multiply them in either order. However, AB and BA are different: i.e., *matrix multiplication is not commutative.*

$$AB = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 7 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} -2(4) + 3(5) & -2(7) + 3(1) \\ 1(4) + 2(5) & 1(7) + 2(1) \end{bmatrix} = \begin{bmatrix} 7 & -11 \\ 14 & 9 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & 7 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4(-2) + 7(1) & 4(3) + 7(2) \\ 5(-2) + 1(1) & 5(3) + 1(2) \end{bmatrix} = \begin{bmatrix} -1 & 26 \\ -9 & 17 \end{bmatrix}$$

(4) Let $A = \begin{bmatrix} 1 & 4 \\ -3 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}$. Find AB and BA (if they make sense).

Solution: AB does not makes sense, because we can't multiply a 3×2 matrix by a 1×3 matrix - but BA does make sense, as we can multiply a 1×3 matrix by a 3×2 matrix.

$$BA = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -3 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (0(1) + 2(-3) + 1(0)) & (0(4) + 2(1) + 1(0)) \end{bmatrix} = \begin{bmatrix} -6 & 2 \end{bmatrix}$$

Another way to view it is that we have a map $\mathbb{R}^1 \xrightarrow{A} \mathbb{R}^3$ and a map $\mathbb{R}^3 \xrightarrow{B} \mathbb{R}^1$, and it only makes sense if we compose these in the order $B \circ A$.

(5) Let $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find AB and BA (if they make sense).

Solution: Both products make sense, but they are different sizes!

$$AB = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = [1(3) + 2(4)] = 11$$
$$BA = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) \\ 4(1) & 4(2) \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix}$$

Notice here that the matrix product of a row vector with a column vector is just the *dot product* in disguise. Its output is a 1×1 matrix, which is just a scalar transformation $\mathbb{R} \to \mathbb{R}$ - i.e., a number.

(6) For any n, let I_n denote the matrix with 1's along the main diagonal and zeroes everywhere else: this is called the *identity matrix.* If A is an $n \times m$ matrix, what is $I_n A$? How about AI_m?

Solution: $I_n A$ is $(n \times n) \cdot (n \times m) = (n \times m)$ - careful checking yields that this is the matrix A. Similarly, AI_m is $(n \times m) \cdot (m \times m) = (n \times m)$ and this multiplication also yields the matrix A. Another way to see this is that the matrix I_n is just the *identity map* from $\mathbb{R}^n \to \mathbb{R}^n$, because if you look at its columns (a few examples given below), it is defined by the fact that $I_n e_1 = e_1, I_n e_2 = e_2, \ldots, I_n e_n = e_n$.

$$I_1 = [1] \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \cdots$$

(7) In 2(b), you found the matrix A for projection onto the line $y = 3x$. What is A^2?

Solution:

$$A^2 = A \cdot A = \frac{1}{100} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = \frac{1}{100} \begin{bmatrix} 10 & 30 \\ 30 & 90 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} = A$$

Geometrically, this makes sense: after you project onto a line, if you do so again, it does nothing. That is, projections are *idempotent.*