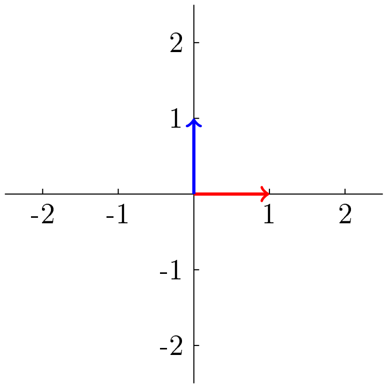
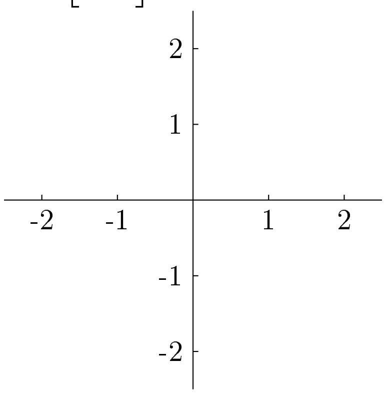
MATH 21B, FEBRUARY 7: ROTATIONS, REFLECTIONS, DILATIONS, PROJECTIONS, AND SHEARS

(1) Consider the vectors $\vec{e}_1$ and $\vec{e}_2$ in $\mathbb{R}^2$.

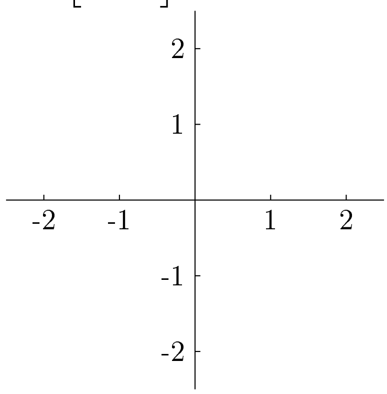


In each part below, you are given a matrix A. Draw what happens to the vectors $\vec{e}_1$ and $\vec{e}_2$ after applying the linear transformation $T(\vec{x}) = A\vec{x}$. Describe the effect of the linear transformation in words.

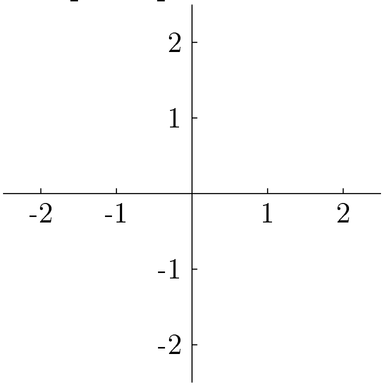
(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$



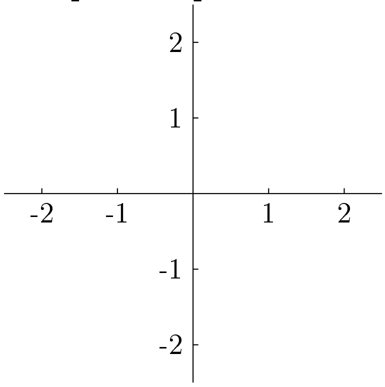
(b) $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$



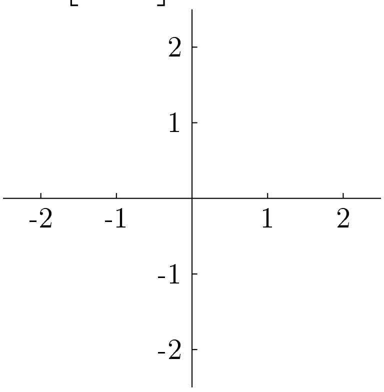
(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$



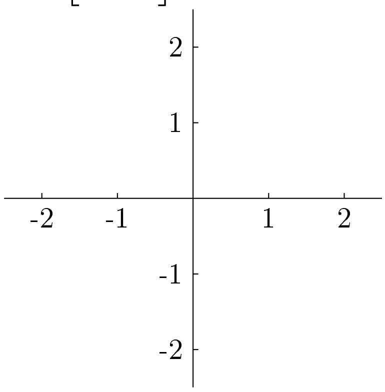
(e) $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$



(d) $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$



(f) $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$



(2) For each of the following linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, write down its matrix.

(a) Rotation counterclockwise by an angle θ around the origin. (Where does this transformation send the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?)

(b) Orthogonal projection onto the line $y = 3x$.

(c) Orthogonal projection onto the line $y = 3x$, followed by dilation by a factor of 2 (i.e. double the length of all vectors).

Given an $n \times p$ matrix A, and a $p \times m$ matrix B, we define the *matrix product* $A \cdot B$ as follows. If B has column vectors

$$B = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \end{bmatrix}$$

then $A \cdot B$ (or just AB) is the $n \times m$ matrix

$$AB = \begin{bmatrix} A\vec{v}_1 & A\vec{v}_2 & \cdots & A\vec{v}_m \end{bmatrix}$$

A represents a linear transformation $\mathbb{R}^p \to \mathbb{R}^n$, and B represents a linear transformation $\mathbb{R}^m \to \mathbb{R}^p$; $A \cdot B$ represents the *composition*, which is a linear transformation $\mathbb{R}^m \to \mathbb{R}^n$.

(3) Let $A = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 7 \\ 5 & 1 \end{bmatrix}$. Find AB and BA (if they make sense).

(4) Let $A = \begin{bmatrix} 1 & 4 \\ -3 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}$. Find AB and BA (if they make sense).

(5) Let $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Find AB and BA (if they make sense).

(6) For any n, let I_n denote the matrix with 1's along the main diagonal and zeroes everywhere else: this is called the *identity matrix*. If A is an $n \times m$ matrix, what is $I_n A$? How about AI_m?

(7) In 2(b), you found the matrix A for projection onto the line $y = 3x$. What is A^2?