MATH 21B, FEBRUARY 2: LINEAR TRANSFORMATIONS

Note: I have changed the names of a few letters and variables in these solutions.

A *linear transformation* is a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfying two properties:
(1) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for any $\vec{x}, \vec{y} \in \mathbb{R}^n$. (Additivity)
(2) $T(\lambda \vec{x}) = \lambda \cdot T(\vec{x})$ for any $\vec{x} \in \mathbb{R}^n$ and any $\lambda \in \mathbb{R}$ (Scaling)

Two important properties that emerge from the above are the following:

- $T(\lambda_1 \vec{x} + \lambda_2 \vec{y}) = \lambda_1 T(\vec{x}) + \lambda_2 T(\vec{y})$. This arises from combining the two properties above.
- $T(\vec{0}) = \vec{0}$, where $\vec{0}$ in any space denotes the all-zeroes vector. This comes from setting $\lambda = 0$ in the second property above.

An $m \times n$ matrix A defines a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ via the *matrix product*:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \end{bmatrix}$$

or, more succinctly, $A\vec{x} = \vec{b}$.

I claim above that any matrix defines a linear transformation, but I haven't explained why this is true. Here, as an example, I'll verify it for any 2×2 matrix, i.e. show that any 2×2 matrix defines a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$. Suppose that we have a 2×2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, for any vectors $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$,

$$A \cdot (\vec{x} + \vec{y}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} = \begin{bmatrix} a(x_1 + y_1) + b(x_2 + y_2) \\ c(x_1 + y_1) + d(x_2 + y_2) \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} + \begin{bmatrix} ay_1 + by_2 \\ cy_1 + dy_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A \cdot \vec{x} + A \cdot \vec{y}$$

which confirms that the first property holds true. Note the structure of the argument here. In practice, I might start by picking a bunch of particular vectors $\vec{x}, \vec{y}$ and verifying that $A \cdot (\vec{x} + \vec{y}) = A \cdot \vec{x} + A \cdot \vec{y}$ holds true. I'd then see that in each such case, the structure of the equations looks somewhat similar. The equations I've written above are the *general case* of that argument - they hold true no matter which vectors $\vec{x}, \vec{y}$ you pick. Checking the second property is similar. For any vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and any constant λ,

$$A \cdot (\lambda \vec{x}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix} = \begin{bmatrix} a\lambda x_1 + b\lambda x_2 \\ c\lambda x_1 + d\lambda x_2 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} = \lambda \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \lambda (A \cdot \vec{x})$$

(1) In each of the following decide whether the given function is a linear transformation. (Added challenge: if it is linear, can you find a matrix which describes it?)

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = 3\vec{x}$

Solution: T is linear, because it is described by the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

$$T(\vec{x}) = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix} = 3\vec{x}$$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Solution: T is not linear, because

$$T(\vec{0}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

which is not the zero vector.

(c) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = xyz$.

Solution: T is not linear, because the scaling property does not hold. For example, if c is any constant,

$$T\left(c \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = T\left(\begin{bmatrix} cx \\ cy \\ cz \end{bmatrix} \right) = (cx) \cdot (cy) \cdot (cz) = c^3 \cdot (xyz)$$

which is not equal to $c \cdot T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = c \cdot (xyz)$.

(d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 - x_2 \end{bmatrix}$.

Solution: T is linear, because it is described by the matrix $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, i.e.

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ 2x_1 - x_2 \end{bmatrix}$$

Note that the two columns of the matrix, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, can be immediately read off as the coefficients of x_1 and x_2 on the right side of our original equation.

(e) $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $T(\vec{x}) = \vec{0}$.

Solution: T linear because it is described by the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. (This is a 4×2 matrix because it maps from $\mathbb{R}^2 \to \mathbb{R}^4$.)

(f) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x})$ = the vector obtained by rotating $\vec{x}$ by 90° counterclockwise around the origin.

Solution: T is linear. We can prove this abstractly, by checking the two required properties, but we can also just generate the 2×2 matrix for T by seeing where it sends the standard basis vectors. Thinking geometrically about what a $90°$ counterclockwise rotation does, we find that

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix} \qquad\qquad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix}$$

Thus, $\begin{bmatrix}0\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\0\end{bmatrix}$ are the columns of the matrix describing T, i.e.

$$T(\vec{x}) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} \cdot \vec{x}$$

(As a sanity check, try plugging in $\vec{x} = \begin{bmatrix}5\\3\end{bmatrix}$. What do you get? If you plot this in the plane, is it what you expect?)

(2) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and all you know about it is that $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}2\\5\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix}$.

(a) Can you compute $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$? If so, find it; if not, explain why not.

Solution: Yes, we can. Because T is linear, we have that for any λ_1 and λ_2,

$$T\left(\lambda_1\begin{bmatrix}1\\2\end{bmatrix} + \lambda_2\begin{bmatrix}2\\5\end{bmatrix}\right) = \lambda_1 T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) + \lambda_2 T\left(\begin{bmatrix}2\\5\end{bmatrix}\right) = \lambda_1\begin{bmatrix}3\\1\end{bmatrix} + \lambda_2\begin{bmatrix}3\\1\end{bmatrix}$$

So we should try to find λ_1, λ_2 such that $\lambda_1\begin{bmatrix}1\\2\end{bmatrix} + \lambda_2\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$. That is, we must solve the system of linear equations

$$\left|\begin{array}{ccccc}\lambda_1 & + & 2\lambda_2 & = & 1\\ 2\lambda_1 & + & 5\lambda_2 & = & 0\end{array}\right|$$

Solve to get $\lambda_1 = 5, \lambda_2 = -2$. Therefore,

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = T\left(5\begin{bmatrix}1\\2\end{bmatrix} - 2\begin{bmatrix}2\\5\end{bmatrix}\right) = 5\begin{bmatrix}3\\1\end{bmatrix} - 2\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}9\\3\end{bmatrix}$$

(b) Can you compute $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$? If so, find it; if not, explain why not.

Solution: Similarly, we want to find λ_1, λ_2 such that $\lambda_1\begin{bmatrix}1\\2\end{bmatrix} + \lambda_2\begin{bmatrix}2\\5\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$. Routine calculation yields $\lambda_1 = -2$, $\lambda_2 = 1$. Therefore,

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = T\left(-2\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}2\\5\end{bmatrix}\right) = -2\begin{bmatrix}3\\1\end{bmatrix} + \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}-3\\1\end{bmatrix}$$

(c) Can you write down a matrix that describes T?

Solution: Yes, we have just calculated the two columns of the matrix that describes T. The matrix is $\begin{bmatrix}9 & -3\\3 & 1\end{bmatrix}$.

(3) Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and all you know about it is that $T\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-2\end{bmatrix}$ and $T\left(\begin{bmatrix}2\\4\\6\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$.

(a) Can you compute $T\left(\begin{bmatrix}1\\3\\5\end{bmatrix}\right)$? If so, find it; if not, explain why not.

Solution: We do not have enough information, because $\begin{bmatrix}1\\3\\5\end{bmatrix}$ cannot be written in the form $\lambda_1 \begin{bmatrix}1\\2\\1\end{bmatrix} + \lambda_2 \begin{bmatrix}2\\4\\6\end{bmatrix}$, i.e., as a *linear combination* of those two vectors. We can see this by observing that in any such linear combination, the second coordinate will be twice the first coordinate.

(b) Can you compute $T\left(\begin{bmatrix}1\\2\\7\end{bmatrix}\right)$? If so, find it; if not, explain why not.

Solution: Yes, we can, because $\begin{bmatrix}1\\2\\7\end{bmatrix} = -2\begin{bmatrix}1\\2\\1\end{bmatrix} + \frac{3}{2}\begin{bmatrix}2\\4\\6\end{bmatrix}$, and therefore,

$$T\left(\begin{bmatrix}1\\2\\7\end{bmatrix}\right) = -2\begin{bmatrix}1\\-2\end{bmatrix} + (3/2)\begin{bmatrix}2\\0\end{bmatrix} = \begin{bmatrix}-2\\4\end{bmatrix} + \begin{bmatrix}3\\0\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$$

Note: In the above question, how much information about the matrix of T can we deduce from the given information? If we write this matrix as $\begin{bmatrix}a_{1,1} & a_{1,2} & a_{1,3}\\ a_{2,1} & a_{2,2} & a_{2,3}\end{bmatrix}$, with six variables that we need to solve for, then the two pieces of information given in the problem tell us

$$\left|\begin{array}{rcrcrcr} a_{1,1} & + & 2a_{1,2} & + & a_{1,3} & = & 1\\ a_{2,1} & + & 2a_{2,2} & + & a_{2,2} & = & -2\\ 2a_{1,1} & + & 4a_{1,2} & + & 6a_{1,3} & = & 2\\ 2a_{2,1} & + & 4a_{2,2} & + & 6a_{2,2} & = & 0 \end{array}\right|$$

Pairing off the first and third row, as well as the second and fourth, we get two systems

$$\left|\begin{array}{rcrcrcr} a_{1,1} & + & 2a_{1,2} & + & a_{1,3} & = & 1\\ 2a_{1,1} & + & 4a_{1,2} & + & 6a_{1,3} & = & 2 \end{array}\right| \qquad \left|\begin{array}{rcrcrcr} a_{2,1} & + & 2a_{2,2} & + & a_{2,2} & = & -2\\ 2a_{2,1} & + & 4a_{2,2} & + & 6a_{2,2} & = & 0 \end{array}\right|$$

with each being a system of two equations in three variables. So, if we knew the value of T at one more vector, it would give us another equation in each of these systems, and if that new equation were *independent* of the equations we have so far (i.e., not a linear combination of the equations we have), then we could solve for T. This corresponds to knowing T at a vector which is not a linear combination of the vectors $\begin{bmatrix}1\\2\\1\end{bmatrix}$ and $\begin{bmatrix}2\\4\\6\end{bmatrix}$.