MATH 21B, FEBRUARY 2: LINEAR TRANSFORMATIONS

A *linear transformation* is a mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfying two properties:

(1) $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ for any $\vec{x}, \vec{y} \in \mathbb{R}^n$.

(2) $T(c\vec{x}) = C \cdot T(\vec{x})$ for any $\vec{x} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$

An $m \times n$ matrix A defines a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ via the *matrix product*:

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \end{bmatrix}$$

or, more succinctly, $A\vec{x} = \vec{b}$.

(1) In each of the following decide whether the given function is a linear transformation. (Added challenge: if it is linear, can you find a matrix which describes it?)

(a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = 3\vec{x}$

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = [xyz]$.

(d) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 2x_1 - x_2 \end{bmatrix}$.

(e) $T : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $T(\vec{x}) = \vec{0}$.

(f) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x})$ = the vector obtained by rotating $\vec{x}$ by $90°$ counterclockwise around the origin.

(2) Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and all you know about it is that $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}2\\5\end{bmatrix}\right) = \begin{bmatrix}3\\1\end{bmatrix}$.

(a) Can you compute $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$? If so, find it; if not, explain why not.

(b) Can you compute $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$? If so, find it; if not, explain why not.

(c) Can you write down a matrix that describes T?

(3) Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, and all you know about it is that $T\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-2\end{bmatrix}$ and $T\left(\begin{bmatrix}2\\4\\6\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix}$.

(a) Can you compute $T\left(\begin{bmatrix}1\\3\\5\end{bmatrix}\right)$? If so, find it; if not, explain why not.

(b) Can you compute $T\left(\begin{bmatrix}1\\2\\7\end{bmatrix}\right)$? If so, find it; if not, explain why not.