MATH 21B, JANUARY 31: MATRICES - ROWS, RANK, AND REDUCED ROW ECHELON FORM

A matrix is said to be in **reduced row echelon form** if it satisfies the following properties:

(1) If a row contains nonzero entries, then the first nonzero entry is a 1, and is called a **leading 1**.
(2) If a column contains a leading 1, then the other entries in that column are 0.
(3) If a row has a leading 1, then every row above it has a leading 1 somewhere to the left.

The number of leading 1's is called the **rank**. Pictorially, a matrix in reduced row echelon form looks something like the following.

$$\begin{bmatrix} 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

where the *'s can be any numbers, and the 1's shown are leading 1's.

(1) In the following systems, use Gauss-Jordan elimination (row operations) to reduce the coefficient matrix to reduced row echelon form. Here, $\vec{x}$ is a *column vector* whose size is equal to the number of variables of the system. How can we then use this form to find all solutions? (Bonus: Can you see a relation between the *rank* of the system and the structure of the solutions?)

(a) $\begin{bmatrix} 1 & -2 & -1 \\ 2 & -4 & -2 \\ 2 & -5 & -4 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$ **Solution:** We use the first row to cancel out all entries below it in the first column.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -4 & -2 & 4 \\ 2 & -5 & -4 & 0 \end{array}\right] - \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & -4 & -2 & 4 \\ 2 & -4 & -2 & 4 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -4 \end{array}\right]$$

We now multiply the third row by -1, and swap it with the second row. We then use the leading 1 (after negation) in this row to cancel out all other nonzero entries in the second column.

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{ccc|c} 0 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This equation now tells us that $\boxed{x_1 = 10 - 3x_3}$ and $\boxed{x_2 = 4 - 2x_3}$. Thus, x_3 is a *free variable*, and it uniquely determines x_1 and x_2 (the variables corresponding to the leading 1's). The general solution is thus $\begin{bmatrix} 10 - 3x_3 & 4 - 2x_3 & x_3 \end{bmatrix}$.

(b) $\begin{bmatrix} 0 & 1 & 2 & 2 & -2 \\ 1 & 0 & 3 & 0 & 4 \\ -1 & 3 & 3 & 0 & -10 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$ **Solution:** First, use the leading entry in the second row to cancel out the -1 in the third row, and then swap this row with the first row.

Then use the leading 1 in the second row to cancel out all other entries in the second column. Finally, scale the third row so that its leading nonzero entry is a 1, and use this 1 to cancel out all other nonzero entries in that column.

$$\left[\begin{array}{ccccc|c} 0 & 1 & 2 & 2 & -2 & 1 \\ 1 & 0 & 3 & 0 & 4 & 5 \\ -1 & 3 & 3 & 0 & -10 & 4 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 4 & 5 \\ 0 & 1 & 2 & 2 & -2 & 1 \\ 0 & 3 & 6 & 0 & -6 & 9 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 4 & 5 \\ 0 & 1 & 2 & 2 & -2 & 1 \\ 0 & 3 & 6 & 0 & -6 & 9 \end{array}\right]$$

$$\to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 4 & 5 \\ 0 & 1 & 2 & 2 & -2 & 1 \\ 0 & 0 & 0 & -6 & 0 & 6 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 4 & 5 \\ 0 & 1 & 2 & 2 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 3 & 0 & 4 & 5 \\ 0 & 1 & 2 & 0 & -2 & 3 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{array}\right]$$

This final system tells us that

$$x_1 = 5 - 3x_3 - 4x_5 \qquad x_2 = 3 - 2x_3 + 2x_5 \qquad x_4 = -1$$

The free variables are x_4 and x_5, and the variables corresponding to the leading 1's are determined by the values of the free variables. The general solution is therefore $\begin{bmatrix} 5 - 3x_3 - 4x_5 & 3 - 2x_3 + 2x_5 & x_3 & -1 & x_5 \end{bmatrix}$.

(c) *This problem had a typo as written. As it was written, the problem was just a standard row reduction analogous to (a). Here's the modified version I had intended.*

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & -4 & -2 \\ 2 & -5 & -4 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$$

Solution: Row reduction is completely analogous to that in (a).

$$\left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 2 & -4 & -2 & 5 \\ 2 & -5 & -4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -3 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & -1 & 2 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 10 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last equation says $0 = 1$, which is impossible! Therefore, there are no solutions to the system.

Note: The only difference between (a) and (c) is the second entry in the constant vector $\vec{b}$. This change takes us from *infinitely* many solutions, to *zero* solutions! In fact, if that entry is anything other than 4, we get zero solutions. There is a geometric way to interpret what is happening here: in part (a), we have three planes intersecting in a line. Changing that number 4 translates one of these planes vertically off the line.

(2) Each of the following matrices is the reduced row echelon form of the augmented matrix of an unknown system. How many solutions does the system have? Explain briefly.

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Solution: The first matrix simply tells us that $x_1 = 3$ (and the second and third row are redundant). There are no restrictions on x_2, the free variable, and therefore, there are infinitely many solutions (forming a line).

The second matrix has third equation $0 = 1$, which means it is inconsistent and there are no solutions.

The third matrix says $x_1 = 3, x_2 = -2, x_3 = 3, x_4 = 0$ and the fifth row is redundant. There is exactly one solution.

Note: In general, any system of equations in n variables has either zero solutions, or has a space of solutions that is a *subspace* sitting inside n-dimensional space. For example, if that subspace is a *point*, then there is a unique solution. If it is a *line*, then there are infinitely many solutions, and we have one free variable. If it is a *plane*, then there are infinitely many solutions, and we have two free variables, and so on. We haven't precisely defined terms like 'subspace', but we will get to that soon!

(3) If you perform Gauss-Jordan elimination on a inconsistent system, how do you recognize that the system is inconsistent?

Solution: This happens if and only if the reduced row echelon form of the augmented matrix has a leading 1 in the last column - i.e., there is a row that is all 0's except for the last entry.

(4) If A is an $m \times n$ matrix (height m, width n) such that $A\vec{x} = \vec{b}$ is consistent for every $\vec{b} \in \mathbb{R}^m$, what can you say about $\text{rref}(A)$? ($\text{rref}(A)$ refers to the reduced row echelon form of A)

I claim that this tells us $\text{rref}(A)$ has no row with all zeroes. Suppose, for a contradiction, that $\text{rref}(A)$ has a row with all zeroes. To get $\text{rref}(A)$, we had to perform some sequence of row operations on A. Perform that same sequence of row operations on $[A|\vec{b}]$. It's clear that there exists some *particular* vector $\vec{b}$ such that the resulting row-reduced version of $[A|\vec{b}]$ will have a row with all zeroes except for the last entry nonzero (in fact, this will happen for almost all $\vec{b}$'s). This would imply that the system $A\vec{x} = \vec{b}$ is inconsistent, which is a contradiction! Therefore, our initial assumption was wrong - $\text{rref}(A)$ must have no rows with all zeroes. (This method of argument is called *proof by contradiction* - it's very useful!)

There are a few other nice ways to interpret what's going on here. First of all, if $\text{rref}(A)$ has no rows with all zeroes, this means that every row has a leading 1. In particular, this implies A has rank m. Its rank (the number of leading 1's) is as large as possible, and the space of solutions to $A\vec{x} = \vec{b}$ is as small as possible (it will be $n - m$). There's a geometric way to see this too: each row of $[A|\vec{b}]$ corresponds to a *hyperplane* in $\mathbb{R}^n$ (i.e., an $(n-1)$-dimensional subspace). These hyperplanes all intersect in some way, and changing $\vec{b}$ translates these hyperplanes around. The statement that this system is consistent for all

$\vec{b}$, means that no matter how we translate these hyperplanes around, they will still intersect (in particular, the situation in 1(a) is impossible).

(5) If the reduced row echelon form of a matrix A has a row of all zeroes, what does this imply about the rows of A? If there are two rows of all zeroes?

Solution: The reduced row echelon form is obtained by adding, scaling, and swapping rows. Therefore, if $\text{rref}(A)$ has a row of all zeroes, this means that some row of A can be obtained by a sum of scaled versions of the other rows. We say that this row is a *linear combination* of the other rows. If there are two rows of all zeroes, it means that there are some *two* rows of A which can each be written as a linear combination of the other $m-2$ rows (if there are m rows).

(6) Find all values of a for which the system $\begin{bmatrix} 2 & a \\ 3 & 6 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is consistent.

Solution 1: Row-reduce the augmented matrix.

$$\left[\begin{array}{cc|c} 2 & a & 1 \\ 3 & 6 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & a & 1 \\ 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & a-4 & 1 \\ 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & a-4 & 1 \end{array}\right]$$

If $a \neq 4$, then the second row has a leading 1 in the second entry, and the system is consistent with the unique solution $x_1 = -\frac{2}{a-4}, x_2 = \frac{1}{a-4}$. If $a = 4$, then the second row is 0 0 1 and the system is inconsistent.

Solution 2: *This solution involves more advanced concepts which we will get to. I don't expect you to be able to construct this solution yourself, but I recommend reading it and trying to understand it!* The equation $\begin{bmatrix} 2 & a \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is *equivalent* to the vector equation

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} a \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

That is, we are looking for a linear combination of the vectors $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} a \\ 6 \end{bmatrix}$ which equals $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We see that if $a = 4$, then the second vector is double the first, and so any linear combination of the two will just equal a multiple of $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ - there is therefore no way to obtain $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$! On the other hand, if $a \neq 4$, then

$$x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} a \\ 6 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \left(\begin{bmatrix} a-4 \\ 0 \end{bmatrix} + \right) \begin{bmatrix} 4 \\ 6 \end{bmatrix}) = (x_1 + 2x_2) \begin{bmatrix} 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} a-4 \\ 0 \end{bmatrix}$$

and so if we set $x_1 + 2x_2 = 0$ and $x_2 = \frac{1}{a-4}$, we will have found our solution. Indeed, this can be done by letting $x_1 = -\frac{2}{a-4}$.