Math 21b, April 25: Some more PDEs and Review

1. Find the solution $f(x,t)$ to the inhomogeneous partial differential equation

$$f_t = -f_{xx} + \sin(7t)$$

given the initial condition $f(x,0) = x$. Use the following steps:

- Find the general form of the solution to the *homogeneous* equation using Fourier series in x.
- Solve for the initial Fourier coefficients by finding a sine series for $f(x,0) = x$.
- Find a particular solution $f_{\text{part}}(x,t)$ to the above inhomogeneous equation using the cookbook method, such that this particular solution satisfies $f_{\text{part}}(x,0) = 0$. Trick: find a solution to $f_t = \sin(7t)$, and the just use this particular solution!
- Add these two to get the general form of the solution for the inhomogeneous equation.
- Solve for the initial Fourier coefficients by finding a sine series for $f(x,0) = x$.

Solution: First, we find the general solution to the homogeneous equation $f_t = -f_{xx}$ using Fourier expansion and then plugging into the homogeneous equation

$$f(x,t) = \sum_k b_k(t)\sin(kx) \implies b_k'(t) = k^2 b_k(t)$$

$$\implies b_k(t) = C_k e^{k^2 t}$$

for some constants $C_1, C_2, \ldots$. Then the general solution is $\boxed{f_{\text{hom}}(x,t) = \sum_k C_k e^{k^2 t}\sin(kx)}$.

Next, we find a particular solution to the inhomogeneous equation using the cookbook method. We guess $f(x,t) = A\cos(7t) + B\sin(7t)$ and plug this into $f_t = -f_{xx} + \sin(7t)$ and solve for A, B. Since f depends on only t, we are just solving $f_t = \sin(7t)$, and we get that $A = -1/7, B = 0$. Thus, $f(x,t) = -\frac{1}{7}\cos(7t)$ is a particular solution. We need to have a solution which satisfies $f_{\text{part}}(x,0) = 0$, so we add the constant $1/7$ to our guess to get $\boxed{f_{\text{part}}(x,t) = \frac{1}{7} - \frac{1}{7}\cos(7t)}$.

Therefore, the general solution[1] to the inhomogeneous differential equation is $f(x,t) = \frac{1}{7} - \frac{1}{7}\cos(7t) + \sum_k C_k e^{k^2 t}\sin(kx)$. We now plug in $t = 0$

$$f(x,0) = 0 + \sum_k C_k \sin(kx)$$

and set it equal to the initial condition (with its Fourier coefficients)

$$f(x,0) = x = \sum_k (-1)^{k+1}\frac{2}{k}\sin(kx) \implies C_k = (-1)^{k+1}\frac{2}{k}$$

[1] Subject to the condition that it is zero at the boundaries when $t = 0$.

Therefore, we get the final solution

$$\boxed{f(x,t) = \frac{1}{7} - \frac{1}{7}\cos(7t) + \sum_k (-1)^{k+1}\frac{2}{k}e^{k^2 t}\sin(kx)}$$

2. (HW31 #4) A laundry line is excited by the wind. It satisfies the differential equation

$$u_{tt} = u_{xx} + \cos(t) + \cos(3t)$$

with initial conditions $u(x,0) = 4\sin(5x) + 10\sin(6x)$ and $u_t(x,0) = 0$. Find the function $u(x,t)$ which satisfies the differential equation.

Solution: First, we find the general solution to the homogeneous PDE $u_{tt} = u_{xx}$

$$u(x,t) = \sum_k b_k(t)\sin(kx) \implies b_k''(t) = -k^2 b_k(t)$$

$$\implies b_k(t) = C_k\cos(kt) + D_k\sin(kt)$$

for some constants $C_1, D_1, C_2, D_2, \ldots$. Then the general solution is $\boxed{u_{\text{hom}}(x,t) = \sum_k (C_k\cos(kt) + D_k\sin(kt))\sin(kx)}$.

Next, we find a particular solution using the cookbook method. We first find a particular solution to $u_{tt} = \cos(t)$: $u(x,t) = -\cos(t)$ works. Similarly, for $u_{tt} = \cos(3t)$, $u(t) = -\frac{1}{9}\cos(3t)$ works. Thus $u(t) = -\cos(t) - \frac{1}{9}\cos(3t)$ is a particular solution to the inhomogeneous equation: in fact, this plus ANY linear function $At + B$ is a solution (because the second derivative of a linear function is zero). We must choose u_{part} so that $u_{\text{part}}(0)$ and $u'_{\text{part}}(0)$ are both zero: we see that

$$\boxed{u_{\text{part}}(t) = \frac{10}{9} - \cos(t) - \frac{1}{9}\cos(3t)}$$

works.

Thus, the general solution to the inhomogeneous differential equation is $u(x,t) = \frac{10}{9} - \cos(t) - \frac{1}{9}\cos(3t) + \sum_k (C_k\cos(kt) + D_k\sin(kt))\sin(kx)$. From this general formula for $u(x,t)$, we calculate $u(x,0)$ and $u_t(x,0)$

$$u(x,0) = \frac{10}{9} - \cos(0) - \frac{1}{9}\cos(0) + \sum_k C_k\sin(kx) = \sum_k C_k\sin(kx)$$

$$u_t(x,0) = \sin(0) + \frac{1}{3}\sin(0) + \sum_k kD_k\sin(kx) = \sum_k kD_k\sin(kx)$$

(Note that $u_{\text{part}}(0)$ and $u'_{\text{part}}(0)$ are both zero because we chose them that way! That's why those terms above disappear.) We now set these equal to the initial conditions $u(x,0) = 4\sin(5x) + 10\sin(6x)$ and $u_t(x,0) = 0$ to obtain $C_5 = 4, C_6 = 10$, and all of the rest of the constants are equal to zero. Therefore,

$$u(x,t) = \frac{10}{9} - \cos(t) - \frac{1}{9}\cos(3t) + \sum_k (C_k\cos(kt) + D_k\sin(kt))\sin(kx)$$

$$= \boxed{\frac{10}{9} - \cos(t) - \frac{1}{9}\cos(3t) + 4\cos(5t)\sin(5t) + 10\cos(6t)\sin(6t)}$$