Math 21b Apr 18: Fourier Series and Partial Differential Equations

Reminder: Let $f(x)$ be a piecewise smooth function defined on the interval $[-\pi, \pi]$. Then it can be expressed via its *Fourier series*:

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

Moreover, if f is an *even* function (i.e. $f(-x) = f(x)$), then all of the b_k's are zero, and if f is an *odd* function (i.e. $f(-x) = -f(x)$) then all of the a_k's are zero. **The Fourier series of $f(x)$ is the expression of $f(x)$ in terms of functions which form an *orthonormal eigenbasis* for the operator D^2.**

Parseval's identity comes from computing the length in two ways:

$$\langle f, f \rangle = a_0^2 + \sum_{k=1}^{\infty} a_k^2 + \sum_{k=1}^{\infty} b_k^2$$

1. Recall $f(x) = x$ has Fourier series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k} \sin(kx)$. Apply Parseval's identity: what identity do you get?

   Solution: You get the equation

   $$\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \sum_{k=1}^{\infty} \frac{4}{k^2}$$

   The left hand side is equal to

   $$\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{x^3}{3\pi} |_{-\pi}^{\pi} = \frac{2}{3}\pi^2$$

   Therefore, we get $\sum_{k=1}^{\infty} \frac{4}{k^2} = \frac{2}{3}\pi^2$, or equivalently, $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. This is a famous identity called the *Basel problem*, and has relevance in number theory.

   Today we'll apply Fourier series to find solutions to two example *partial differential equations:* the *heat equation* and the *wave equation.*

2. (The Heat Equation) Let $f(x, t)$ be a function of position $x \in [0, \pi]$ and time $t \geq 0$, governed by the partial differential equation

   $$\frac{\partial}{\partial t} f = \frac{\partial^2}{\partial x^2} f$$

   We abbreviate this as $f_t = f_{xx}$ for brevity.

   (a) Write a Fourier decomposition $f(x, t) = \sum_{k=1}^{\infty} b_k(t) \sin(kx)$ where the coefficients $b_1(t), b_2(t), b_3(t), \ldots$ are varying with time. When you plug this into the heat equation, what differential equations do you get for the functions $b_k(t)$?

Solution: When we plug the Fourier series $f(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin(kx)$ into the differential equation, we get

$$\sum_{k=1}^{\infty} b_k'(t)\sin(kx) = \sum_{k=1}^{\infty} b_k(t)(-k^2)\sin(kx)$$

Both sides here have to have all Fourier coefficients equal, and therefore,

$$b_k'(t) = -k^2 b_k(t) \qquad\qquad k = 1, 2, 3, \ldots$$

(b) Write down a general solution for each $b_k(t)$.

Solution: The general solution is $b_k(t) = C_k e^{-k^2 t}$ where $C_k = b_k(0)$ can be any constant. In particular, this means

$$f(x,0) = \sum_{k=1}^{\infty} C_k \sin(kx) \implies f(x,t) = \sum_{k=1}^{\infty} C_k e^{-k^2 t}\sin(kx)$$

(c) Suppose we are given the initial condition $f(x,0) = \sin(x) - \sin(3x)$. What are the initial Fourier coefficients $b_k(0)$? Use these to write down the solution $f(x,t)$.

Solution: The initial Fourier coefficients $b_k(0)$ are $b_1(0) = 1, b_3(0) = -1$, and $b_k(0) = 0$ for $k = 2, 4, 5, 6, \ldots$. Therefore, the solution is $f(x,t) = e^{-t}\sin(x) - e^{-9t}\sin(3x)$.

Suppose a metal rod of length π is heated uniformly to 50° C, and then its ends are plunged into ice baths at 0° C: you want to determine how the entire rod cools over time. The temperature satisfies the heat equation $f_t = f_{xx}$.

(d) We have the initial condition $f(x,0) = 50$ and boundary conditions $f(0,t) = f(\pi,t) = 0$. Calculate the initial Fourier coefficients $b_k(0)$ and use these to write down the solution $f(x,t)$.

Solution: To calculate the Fourier coefficients of $f(x,0) = 50$, we calculate

$$b_k(0) = \frac{2}{\pi}\int_0^{\pi} 50\sin(kx)dx = -\frac{100}{k\pi}\cos(kx)|_0^{\pi} = \begin{cases} \frac{200}{k\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

That is, $50 = \frac{200}{\pi}\sin(x) + \frac{200}{3\pi}\sin(3x) + \frac{200}{5\pi}\sin(5x) + \ldots$. Thus,

$$f(x,t) = \frac{200}{\pi}e^{-t}\sin(x) + \frac{200}{3\pi}e^{-9t}\sin(3x) + \frac{200}{5\pi}e^{-25t}\sin(5x) + \ldots$$

This can be expressed in sum notation as

$$f(x,t) = \frac{200}{\pi}\sum_{m=0}^{\infty}\frac{1}{2m+1}e^{-(2m+1)^2 t}\sin((2m+1)x)$$

(e) (Adjusting the boundary conditions) Suppose instead of both being plunged into ice baths, the two ends of the metal rod are left alone, i.e. boundary conditions $f(t, 0) = f(t, \pi) = 50$. What happens to the temperature over time?

Solution: The temperature will remain constant throughout the rod at 50° C.

(f) Let f_p be the solution that you found in part (e). Show that if f is any solution to the heat equation satisfying the boundary conditions $f(0, t) = f(\pi, t) = 50$, then $g = f - f_p$ is a solution to the heat equation satisfying $g(0, t) = g(\pi, t) = 0$.

General strategy:

- First normalize the boundary conditions: subtract a linear function f_p from $f(x, 0)$ so that $g = f - f_p$ is equal to 0 at both $x = 0$ and $x = \pi$.
- Fourier decompose g with a sine series, let its Fourier coefficients be $b_1(0), b_2(0), \ldots$.
- The heat equation gives us differential equations for the behavior of $b_k(t)$: solve these to obtain the functions $b_k(t)$, which are the Fourier coefficients of $g(x, t)$. Note that as $t \to \infty$, $g(x, t) \to 0$ for all x.
- Add back on the linear function f_p to obtain the solution $f(x, t)$.

3. (The Wave Equation) Consider a partial differential equation which might model the propagation of a wave through a string:

$$u_{tt} = u_{xx}$$

We will assume that we have boundary conditions $u(0, t) = u(\pi, t) = 0$, i.e. the two ends of the string are held fixed.

(a) As before, write down a Fourier decomposition for $u(x, t)$ in terms of a sine series $\sum_{k=1}^{\infty} b_k(t) \sin(kx)$. What are the differential equations for the $b_k(t)$'s? Write down a general solution for each $b_k(t)$.

Solution: Let $u(x, t) = \sum_{k=1}^{\infty} b_k(t) \sin(kx)$. Then if we plug this into the different equation, we get

$$\sum_{k=1}^{\infty} b_k''(t) \sin(kx) = \sum_{k=1}^{\infty} b_k(t)(-k^2) \sin(kx)$$

$$\implies b_k''(t) = -k^2 b_k(t)$$

This differential equation has the general solution $b_k(t) = C_k \cos(kt) + D_k \sin(kt)$. If we have initial conditions $b_k(0), b_k'(0)$, then we get $b_k(0) = C_k$ and $b_k'(0) = kD_k$, i.e.

$$\boxed{b_k(t) = b_k(0)\cos(kt) + \frac{b_k'(0)}{k}\sin(kt)}$$

(b) Suppose that we have initial conditions $u(x,0) = \sin(5x) + 3\sin(8x)$ and $u_t(x,0) = 2\sin(7x) + \sin(8x)$. Fourier decompose these to find $b_k(0)$ and $b_k'(0)$, and thus solve for $b_k(t)$ and find the solution $u(x,t)$.

Solution: The Fourier decomposition is essentially given by the forms of these functions. Using the boxed formula above, we get

$$b_5(t) = 1 \cdot \cos(5t) + 0 \cdot \sin(5t) = \cos(5t)$$

$$b_7(t) = 0 \cdot \cos(7t) + 2 \cdot \sin(7t) = 2\sin(7t)$$

$$b_8(t) + 3 \cdot \cos(8t) + 1 \cdot \sin(8t) = 3\cos(8t) + \sin(8t)$$

and all other Fourier coefficients $b_k(t)$ are just zero. Therefore,

$$u(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin(kx) = \cos(5t)\sin(5x) + 2\sin(7t)\sin(7x) + (3\cos(8t) + \sin(8t))\sin(8x)$$

4. Find $f(x,t)$ such that

$$f_t = f_{xx} + \cos(t)$$

given initial conditions $f(x,0) = \sin(x) - \sin(3x)$. (Hint: first find a particular solution f_{part} which depends only on t, and find the general solution f_{hom} to the homogeneous equation $f_t = f_{xx}$. Then find the correct sum of these which gives the given initial conditions.)

Solution: A particular solution to this equation is $f(x,t) = \sin(t)$: its partial derivative with respect to t is $\cos(t)$ and with respect to x is zero (because the function has no x-dependence). We already know the general solution to the homogeneous equation $f_t = f_{xx}$ is

$$f_{\text{hom}}(x,t) = \sum_{k=1}^{\infty} C_k e^{-k^2 t}\sin(kx)$$

where $C_1, C_2, \ldots$ are arbitrary constants. Therefore, the general solution to the nonhomogeneous equation above is

$$f(x,t) = \sin(t) + \sum_{k=1}^{\infty} C_k e^{-k^2 t}\sin(kx)$$

Set $t = 0$ and set this equal to the initial condition $f(x,0)$.

$$0 + \sum_{k=1}^{\infty} C_k \cdot 1 \cdot \sin(kx) = \sin(x) - \sin(3x)$$

and therefore $C_1 = 1, C_3 = -1$, and $C_k = 0$ for $k = 2, 4, 5, \ldots$. So the desired solution is

$$f(x,t) = \sin(t) + \sin(x) - \sin(3x)$$

MORE EXAMPLE PROBLEMS BELOW

5. Find the solution to the heat equation

$$f_t = 5f_{xx}$$

given the initial condition $f(x,0) = \cos(x)$ on $0 \leq x \leq \pi$.

Solution: First, Fourier expand $f(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin(kx)$, and plug this into the differential equation. We get

$$b_k'(t) = -5k^2 b_k(t) \implies \boxed{b_k(t) = b_k(0)e^{-5k^2 t}}$$

Now we find the Fourier coefficients of $f(x,0)$

$$C_k = \frac{2}{\pi}\int_0^{\pi} \cos(x)\sin(kx)dx = \frac{1}{\pi}\int_0^{\pi}(\sin((k+1)x) + \sin((k-1)x))dx$$

this is using the trig identities $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ and $\sin(A-B) = \sin(A)\cos(B) - \cos(A)\sin(B)$.

$$= \frac{1}{\pi}\left(-\frac{\cos((k+1)x)}{k+1} - \frac{\cos((k-1)x)}{k-1}\right)\Big|_0^{\pi}$$

When k is odd, this is zero, and when k is even, it equals

$$= \frac{2}{\pi}\left(\frac{1}{k+1} + \frac{1}{k-1}\right) = \frac{4k}{(k^2-1)\pi}$$

Thus, $f(x,0) = \cos(x)$ has the sine series

$$f(x,0) = \cos(x) = \frac{8}{3\pi}\sin(2x) + \frac{16}{15\pi}\sin(4x) + \frac{24}{35\pi}\sin(6x) + \ldots = \sum_{k \text{ even}} \frac{4k}{(k^2-1)\pi}\sin(kx)$$

$$= \sum_{m=1}^{\infty} \frac{8m}{(4m^2-1)\pi}\sin(2mx)$$

It follows that $f(x,t)$ has the sine series

$$f(x,t) = \frac{8}{3\pi}e^{-20t}\sin(2x) + \frac{16}{15\pi}e^{-80t}\sin(4x) + \ldots = \sum_{k \text{ even}} \frac{4k}{(k^2-1)\pi}e^{-5k^2 t}\sin(kx)$$

$$= \boxed{\sum_{m=1}^{\infty} \frac{8m}{(4m^2-1)\pi}e^{-20m^2 t}\sin(2mx)}$$

6. Find the general solution to the equation

$$u_{tt} = u_{xxxxxx} + 2u_{xxxx} + u_{xx}$$

Solution: Fourier expand $u(x,t) = \sum_{k=1}^{\infty} b_k(t)\sin(kx)$, and plug this into the differential equation.

$$b_k''(t) = (-k^6 + 2k^4 - k^2)b_k(t) \implies b_k(t) = C_k \cos(\sqrt{k^6 - 2k^4 + k^2}x) + D_k \sin(\sqrt{k^6 - 2k^4 + k^2}x)$$

Since $\sqrt{k^6 - 2k^4 + k^2} = \pm k(k^2 - 1)$, this means $b_k(t) = C_k \cos(k(k^2-1)x) + D_k \sin(k(k^2-1)x)$.

(Now, if we were given initial conditions $u(x,0)$ and $u_t(x,0)$, we could find their respective Fourier expansions

$$u(x,0) = \sum_k b_k(0)\sin(kx)$$

$$u_t(x,0) = \sum_k b_k'(0)\sin(kx)$$

and use the values of $b_k(0), b_k'(0)$ to calculate the coefficients C_k, D_k for every k.)