Math 21b Apr 18: Fourier Series and Partial Differential Equations

Reminder: Let $f(x)$ be a piecewise smooth function defined on the interval $[-\pi, \pi]$. Then it can be expressed via its *Fourier series*:

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos(kx) + \sum_{k=1}^{\infty} b_k \sin(kx)$$

Moreover, if f is an *even* function (i.e. $f(-x) = f(x)$), then all of the b_k's are zero, and if f is an *odd* function (i.e. $f(-x) = -f(x)$) then all of the a_k's are zero. **The Fourier series of $f(x)$ is the expression of $f(x)$ in terms of functions which form an *orthonormal eigenbasis* for the operator D^2.**

Parseval's identity comes from computing the length in two ways:

$$\langle f, f \rangle = a_0^2 + \sum_{k=1}^{\infty} a_k^2 + \sum_{k=1}^{\infty} b_k^2$$

1. Recall $f(x) = x$ has Fourier series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k} \sin(kx)$. Apply Parseval's identity: what identity do you get?

Today we'll apply Fourier series to find solutions to two example *partial differential equations:* the *heat equation* and the *wave equation.*

2. (The Heat Equation) Let $f(x, t)$ be a function of position $x \in [0, \pi]$ and time $t \geq 0$, governed by the partial differential equation

$$\frac{\partial}{\partial t} f = \frac{\partial^2}{\partial x^2} f$$

We abbreviate this as $f_t = f_{xx}$ for brevity.

(a) Write a Fourier decomposition $f(x, t) = \sum_{k=1}^{\infty} b_k(t) \sin(kx)$ where the coefficients $b_1(t), b_2(t), b_3(t), \ldots$ are varying with time. When you plug this into the heat equation, what differential equations do you get for the functions $b_k(t)$?

(b) Write down a general solution for each $b_k(t)$.

(c) Suppose we are given the initial condition $f(x,0) = \sin(x) - \sin(3x)$. What are the initial Fourier coefficients $b_k(0)$? Use these to write down the solution $f(x,t)$.

Suppose a metal rod of length π is heated uniformly to 50° C, and then its ends are plunged into ice baths at 0° C: you want to determine how the entire rod cools over time. The temperature satisfies the heat equation $f_t = f_{xx}$.

(d) We have the initial condition $f(x,0) = 50$ and boundary conditions $f(0,t) = f(\pi,t) = 0$. Calculate the initial Fourier coefficients $b_k(0)$ and use these to write down the solution $f(x,t)$.

(e) (Adjusting the boundary conditions) Suppose instead of both being plunged into ice baths, the two ends of the metal rod are left alone, i.e. boundary conditions $f(t,0) = f(t,\pi) = 50$. What happens to the temperature over time?

(f) Let f_p be the solution that you found in part (e). Show that if f is any solution to the heat equation satisfying the boundary conditions $f(0,t) = f(\pi,t) = 50$, then $g = f - f_p$ is a solution to the heat equation satisfying $g(0,t) = g(\pi,t) = 0$.

General strategy:

- First normalize the boundary conditions: subtract a linear function f_p from $f(x,0)$ so that $g = f - f_p$ is equal to 0 at both $x = 0$ and $x = \pi$.
- Fourier decompose g with a sine series, let its Fourier coefficients be $b_1(0), b_2(0), \ldots$.
- The heat equation gives us differential equations for the behavior of $b_k(t)$: solve these to obtain the functions $b_k(t)$, which are the Fourier coefficients of $g(x,t)$. Note that as $t \to \infty$, $g(x,t) \to 0$ for all x.
- Add back on the linear function f_p to obtain the solution $f(x,t)$.

3. (The Wave Equation) Consider a partial differential equation which might model the propagation of a wave through a string:

$$u_{tt} = u_{xx}$$

We will assume that we have boundary conditions $u(0,t) = u(\pi,t) = 0$, i.e. the two ends of the string are held fixed.

(a) As before, write down a Fourier decomposition for $u(x,t)$ in terms of a sine series $\sum_{k=1}^{\infty} b_k(t)\sin(kx)$. What are the differential equations for the $b_k(t)$'s? Write down a general solution for each $b_k(t)$.

(b) Suppose that we have initial conditions $u(x,0) = \sin(5x) + 3\sin(8x)$ and $u_t(x,0) = 2\sin(7x) + \sin(8x)$. Fourier decompose these to find $b_k(0)$ and $b_k'(0)$, and thus solve for $b_k(t)$ and find the solution $u(x,t)$.

4. Find $f(x,t)$ such that

$$f_t = f_{xx} + \cos(t)$$

given initial conditions $f(x,0) = \sin(x) - \sin(3x)$. (Hint: first find a particular solution f_{part} which depends only on t, and find the general solution f_{hom} to the homogeneous equation $f_t = f_{xx}$. Then find the correct sum of these which gives the given initial conditions.)