Math 21b Apr 18: The Fourier Transform

Today, we'll be studying the linear space C^∞_{per} of 2π-periodic functions. Such functions are completely determined by knowing their value on the interval $[-\pi, \pi]$, so we will sometimes just think of these as functions defined on the interval $[-\pi, \pi]$.

1. The *inner product* of two functions f and g in C^∞_{per} is defined to be

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$$

Compute the inner product $\langle \cos(x), \sin(x)\rangle$.

Solution: $\langle \cos(x), \sin(x)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(x)\sin(x)dx$. Since $\cos(x)$ is an *even* function and $\sin(x)$ is an *odd* function, $\cos(x)\sin(x)$ is an odd function and therefore $\int_{-a}^{a} \cos(x)\sin(x)dx = 0$ for any a. Hence, $\langle \cos(x), \sin(x)\rangle = 0$.

2. The *length* of a function $f(x)$ is defined to be $||f|| = \sqrt{\langle f, f\rangle}$.

   (a) Compute $||\cos(x)||$ and $||\sin(x)||$.

   Solution:

$$\langle \cos(x), \cos(x)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(x)dx \qquad \langle \sin(x), \sin(x)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(x)dx$$

   These two integrals must be equal, because $\cos(x) = \sin(x+\pi/2)$. But since $\cos^2(x)+\sin^2(x) = 1$,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2(x)dx + \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} 1dx = \frac{1}{\pi}(2\pi) = 2$$

   Therefore, $\langle \cos(x), \cos(x)\rangle = \langle \sin(x), \sin(x)\rangle = 1$, and so both functions have length 1.

   (b) What about $||\cos(kx)||$ and $||\sin(kx)||$, for k a positive integer? (Hint: this should be quite quick from the previous calculation.)

   Solution: We similarly use the fact that $\cos^2(kx) + \sin^2(kx) = 1$ to obtain (via an argument identical to that above) that $\cos(kx)$ and $\sin(kx)$ have length 1.

A collection of functions $f_1, \ldots, f_n$ is called *orthonormal* if they satisfy two properties:

- Pairwise orthogonal, i.e. $\langle f_i, f_j \rangle = 0$ $(i \neq j)$.
- Length 1, i.e. $\langle f_i, f_i \rangle = 1$ for each i.

3. Argue that we've shown the functions $1/\sqrt{2}, \cos(x), \sin(x)$ are orthonormal.

Solution: In (1), we showed that $\cos(x)$ and $\sin(x)$ are orthogonal, and in (2), we showed that they have unit length. So we just have to argue that the constant function $1/\sqrt{2}$ has unit length and is orthogonal to both $\cos(x)$ and $\sin(x)$. This calculation is below.

$$\langle 1/\sqrt{2}, 1/\sqrt{2} \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} (1/\sqrt{2})^2 dx = \frac{1}{\pi}(2\pi)(1/2) = 1$$

$$\langle 1/\sqrt{2}, \cos(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos(x)}{\sqrt{2}} dx = 0 \qquad \langle 1/\sqrt{2}, \sin(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\sin(x)}{\sqrt{2}} dx = 0$$

these last two are because the positive parts of cosine cancel out the negative parts in the integral, same for sine.

4. Let a, b be positive integers. Can you calculate $\langle \sin(ax), \cos(bx) \rangle$?

Solution:

$$\langle \sin(ax), \cos(bx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(ax)\cos(bx) dx$$

Since $\sin(ax)$ is odd and $\cos(bx)$ is even, $\sin(ax)\cos(bx)$ is odd and therefore the integral above is zero. Therefore, $\sin(ax)$ and $\cos(bx)$ are orthogonal, for any a, b. More generally, **any odd function is orthogonal to any even function.**

Fact: The functions

$$1/\sqrt{2}, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \ldots$$

are orthonormal.

Proof: We have already proven in (2) that all of these functions are of unit length, so we just need to show they're all pairwise orthogonal. It's clear by an argument analogous to that in (3) that the constant function is orthogonal to all of the others. We proved in (4) that every cosine function is orthogonal to every sine function. So all that remains to be shown is that $\langle \cos(ax), \cos(bx) \rangle = 0$ and $\langle \sin(ax), \sin(bx) \rangle = 0$ when $a \neq b$.

$$\langle \cos(ax), \cos(bx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(ax)\cos(bx) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\cos((a+b)x) + \cos((a-b)x)\right) dx$$

If $a \neq b$ then this last integral is equal to zero, because the positive parts cancel the negative parts. This proves the required orthogonality between the different cosine functions. Orthogonality between the sine functions is proven similarly.

5. Recall that if $u_1, \ldots, u_m$ in $\mathbb{R}^n$ is an orthonormal set of vectors, then

$$\text{proj}_V(v) = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + \ldots + (v \cdot u_m)u_m$$

is the projection of v onto the span of $u_1, \ldots, u_m$.

(a) Consider the function $f(x) = x$ defined on the interval $[-\pi, \pi]$. Calculate its projection onto the space spanned by the orthonormal functions $1/\sqrt{2}, \cos(x), \sin(x)$.

Solution: This projection is equal to

$$\langle x, 1/\sqrt{2}\rangle(1/\sqrt{2}) + \langle x, \cos(x)\rangle \cos(x) + \langle x, \sin(x)\rangle \sin(x)$$

Since $f(x) = x$ is odd, $\langle x, 1/\sqrt{2}\rangle = 0$ and $\langle x, \cos(x)\rangle = 0$. We explicitly calculate

$$\langle f(x), \sin(x)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(x)dx$$

Apply integration by parts with $u = x$ and $dv = \sin(x)dx$ to get

$$= -\frac{1}{\pi}x\cos(x)|_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(x)dx$$

The integral here disappears, while the first term evaluates to $-\frac{1}{\pi}(\pi)(-1) + \frac{1}{\pi}(-\pi)(-1) = 2$. Therefore, the desired projection is

$$0(1/\sqrt{2}) + 0\cos(x) + 2\sin(x) = 2\sin(x)$$

(b) In general, let T_n be the space defined by

$$T_n = \text{span}(1/\sqrt{2}, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(nx), \sin(nx))$$

Calculate the projection of $f(x) = x$ onto T_n.

Solution: We clearly only need to compute the inner products $\langle x, \sin(kx)\rangle$ for $k = 1, \ldots, n$, as the other terms disappear.

$$\langle x, \sin(kx)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(kx)dx = -\frac{1}{k\pi}x\cos(kx)|_{-\pi}^{\pi} + \frac{1}{k\pi}\int_{-\pi}^{\pi}\cos(kx)dx$$

$\cos(k\pi) = -1$ if k is odd and $+1$ if k is even. So the expression above evaluates to $\frac{2}{k}$ if k is odd and $-\frac{2}{k}$ if k is even. That is, we get $\langle x, \sin(kx)\rangle = (-1)^{k-1}\frac{2}{k}$. Thus

$$\text{proj}_{T_n}(x) = 2\sin(x) - \frac{2}{2}\sin(2x) + \frac{2}{3}\sin(3x) - \ldots + (-1)^{n-1}\frac{2}{n}\sin(nx) = \sum_{k=1}^{n}(-1)^{k-1}\frac{2}{k}\sin(kx)$$

Fourier series: Let $f(x)$ be a piecewise continuous function defined on $[-\pi, \pi]$. Then it is equal to its *Fourier series*

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where the coefficients are defined by

$$a_0 = \langle f(x), 1/\sqrt{2} \rangle \qquad a_n = \langle f(x), \cos(nx) \rangle \qquad b_n = \langle f(x), \sin(nx) \rangle$$

This expression may be thought of as $\text{proj}_{T_\infty}(f(x))$, the projection of $f(x)$ onto the space of all trigonometric polynomials.

6. Let $f(x)$ be defined on $[-\pi, \pi]$ by

$$f(x) = \begin{cases} 1 & \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Calculate the Fourier coefficients. What is $\text{proj}_{T_{99}}(f(x))$?

Solution: Since $f(x)$ is an *even* function, the sine coefficients b_n are all equal to zero. So we only need to compute $a_0, a_1, a_2, \ldots$. Note by definition that $\langle f(x), g(x) \rangle = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} g(x)dx$. So

$$a_0 = \langle f(x), 1/\sqrt{2} \rangle = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{2}} dx = \frac{1}{\sqrt{2}}$$

$$a_k = \langle f(x), \cos(kx) \rangle = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(kx)dx = \frac{1}{k\pi} \sin(kx)|_{-\pi/2}^{\pi/2}$$

For $k = 1, 2, 3, 4$ we find these go $\frac{2}{\pi}, 0, -\frac{2}{3\pi}, 0, \frac{2}{5\pi}, 0, \ldots$. That is, $f(x)$ has the Fourier series

$$f(x) = \frac{2}{\pi} \cos(x) - \frac{2}{3\pi} \cos(3x) + \frac{2}{5\pi} \cos(5x) - \frac{2}{7\pi} \cos(7x) + \ldots = \sum_{m=0}^{\infty} \frac{2}{(2m+1)\pi} \cos((2m+1)x)$$

The projection onto T_{99} is this sum all the way up to the term $-\frac{2}{99\pi} \cos(99x)$.