Math 21b Apr 18: The Fourier Transform

Today, we'll be studying the linear space C^∞_{per} of 2π-periodic functions. Such functions are completely determined by knowing their value on the interval $[-\pi, \pi]$, so we will sometimes just think of these as functions defined on the interval $[-\pi, \pi]$.

1. The *inner product* of two functions f and g in C^∞_{per} is defined to be

$$\langle f, g\rangle = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)g(x)dx$$

   Compute the inner product $\langle \cos(x), \sin(x)\rangle$.

2. The *length* of a function $f(x)$ is defined to be $||f|| = \sqrt{\langle f, f\rangle}$.

   (a) Compute $||\cos(x)||$ and $||\sin(x)||$.

   (b) What about $||\cos(kx)||$ and $||\sin(kx)||$, for k a positive integer? (Hint: this should be quite quick from the previous calculation.)

A collection of functions $f_1, \ldots, f_n$ is called *orthonormal* if they satisfy two properties:

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3. Argue that we've shown the functions $1/\sqrt{2}, \cos(x), \sin(x)$ are orthonormal.

4. Let a, b be positive integers. Can you calculate $\langle \sin(ax), \cos(bx) \rangle$?

Fact: The functions $$1/\sqrt{2}, \cos(x), \sin(x), \cos(2x), \sin(2x), \cos(3x), \sin(3x), \ldots$$ are orthonormal.

5. Recall that if $u_1, \ldots, u_m$ in $\mathbb{R}^n$ is an orthonormal set of vectors, then

$$\mathrm{proj}_V(v) = (v \cdot u_1)u_1 + (v \cdot u_2)u_2 + \ldots + (v \cdot u_m)u_m$$

is the projection of v onto the span of $u_1, \ldots, u_m$.

(a) Consider the function $f(x) = x$ defined on the interval $[-\pi, \pi]$. Calculate its projection onto the space spanned by the orthonormal functions $1/\sqrt{2}, \cos(x), \sin(x)$.

(b) In general, let T_n be the space defined by

$$T_n = \mathrm{span}(1/\sqrt{2}, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots, \cos(nx), \sin(nx))$$

Calculate the projection of $f(x) = x$ onto T_n.

(extra space)

Fourier series: Let $f(x)$ be a piecewise continuous function defined on $[-\pi, \pi]$. Then it is equal to its *Fourier series*

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where the coefficients are defined by

$$a_0 = \langle f(x), 1/\sqrt{2} \rangle \qquad a_n = \langle f(x), \cos(nx) \rangle \qquad b_n = \langle f(x), \sin(nx) \rangle$$

This expression may be thought of as $\text{proj}_{T_\infty}(f(x))$, the projection of $f(x)$ onto the space of all trigonometric polynomials.

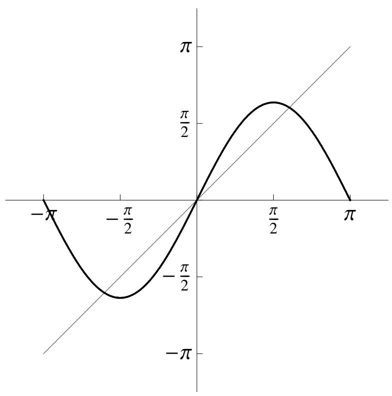
6. Let $f(x)$ be defined on $[-\pi, \pi]$ by

$$f(x) = \begin{cases} 1 & \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

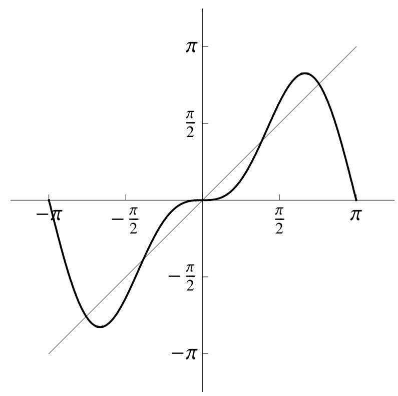
Calculate the Fourier coefficients. What is $\text{proj}_{T_{99}}(f(x))$?

(extra space)

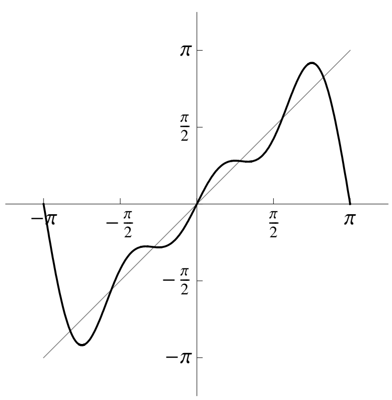
Let $f(x) = x$ (problem 5). Here are the graphs of $f_n = \mathrm{proj}_{T_n} f$ for various values of n.



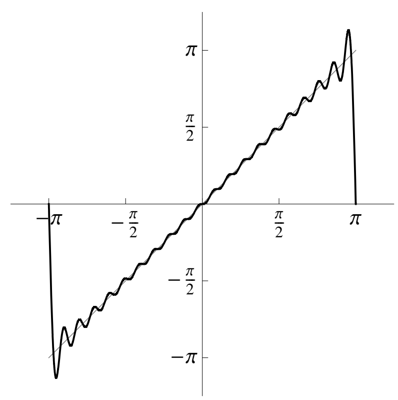
$n = 1$



$n = 2$



$n = 3$



$n = 4$